Linear Algebra I, Sheet 4, MT2019

Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

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P1. Find the column rank of each of the matrices in Q1 on this sheet. What do you notice?

$$\text{(a)} \begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}$$

(a) The column rank is the dimension of the subspace of $\mathbb{R}^3$ spanned by the four columns. By taking the transpose, it is enough to find the row space of the transpose.

We can use our usual technique of applying EROs to put the matrix in echelon form, and we find that the matrix reduces to

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & \frac{11}{6} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so the row rank is 3, and hence the column rank of the original matrix is 3.

(b) Similarly, we use EROs on the transpose. We can reduce the transpose to

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and we find that the column rank is 2.

[We could also have seen directly that the second and third columns are both linear combinations of the first and fourth columns, and the first and fourth columns are linearly independent.]

(c) In this case, we can reduce the transpose to

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so the column rank is 3.

Comparing with the row ranks we found in Q1, we can see that in each of these three cases the row rank is the same as the column rank. Intriguing...

But note that in each case the row space and column space are different: the row space is a subspace of $\mathbb{R}^4$, whereas the column space is a subspace of $\mathbb{R}^3$.

[In lectures, we'll explore whether in general there is a link between row rank and column rank.]

P2. Taking A as each of the three matrices in Q1 on this sheet, find all the solutions to $Ax = 0$ in each case. What do you notice?

$$\text{(a)} \begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}$$

(a) We can use EROs to reduce the matrix to RRE form. In fact we have already put this matrix in echelon form on Sheet 2 (but note that this is not unique, so it's possible you have a different echelon form from the one I give here).

[We normally use an augmented matrix when solving a system of simultaneous linear equations. In this case, the additional column in the matrix would have all entries 0, and these would not change when applying EROs, so we do not record them.]

$$\begin{pmatrix} 2 & 4 & -3 & 0 \\ 1 & -4 & 3 & 0 \\ 3 & -5 & 2 & 1 \end{pmatrix} \xrightarrow[\text{Sheet 2}]{} \begin{pmatrix} 1 & -4 & 3 & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & -\frac{4}{7} \end{pmatrix} \xrightarrow[R_1 \to R_1 + 4R_2]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & -\frac{4}{7} \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + \frac{3}{4} R_3]{} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{7} \\ 0 & 0 & 1 & -\frac{4}{7} \end{pmatrix}.$$

We read off that the set of solutions to $Ax = 0$ is $\text{Span}\{(0\ 3\ 4\ 7)^T\}$.

(b) We use the same strategy again

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{pmatrix} \xrightarrow[\text{Sheet 2}]{} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The set of solutions to $Ax = 0$ is $\text{Span}\{(1\ -2\ 1\ 0)^T, (-2\ 1\ 0\ 1)^T\}$.

(c) And finally, using the same strategy again,

$$\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix} \xrightarrow[\text{Sheet 2}]{} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{} \begin{pmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + 2R_3]{R_1 \to R_1 - 4R_3} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

So the set of solutions to $Ax = 0$ is $\text{Span}\{(1\ -2\ 1\ 0)^T\}$.

We know from Q1 that the matrices in (a) and (c) have row rank 3, and in these two cases the set of solutions to $Ax = 0$ is a space with dimension 1. But the matrix in (b) has row rank 2, and it looks as though the set of solutions to $Ax = 0$ is somehow compensating for the smaller row rank, by being a space with dimension 2.

[This relates to important ideas that we'll explore much more in lectures later this term!]

P3. Let $V = V_1 \oplus V_2$. Take a subspace $U \leqslant V$. Must it be true that $U = U_1 \oplus U_2$ where $U_1 \leqslant V_1$ and $U_2 \leqslant V_2$? (Find a proof or give a counterexample.)

This is not true. Here is a counterexample.

Let $V = \mathbb{R}^2$, and let $V_1 = \text{Span}\{(1\ 0)\} = \{(x\ 0) : x \in \mathbb{R}\}$ and $V_2 = \text{Span}\{(0\ 1)\} = \{(0\ y) : y \in \mathbb{R}\}$. Then $V = V_1 + V_2$ and $V_1 \cap V_2 = \{(0\ 0)\}$, so $V = V_1 \oplus V_2$.

Consider $U = \text{Span}\{(1\ 1)\} = \{(z\ z) : z \in \mathbb{R}\}$. This is certainly a subspace of V (it is the span of a set of vectors, which guarantees that it is a subspace).

But we cannot write $U = U_1 \oplus U_2$ where $U_1 \leqslant V_1$ and $U_2 \leqslant V_2$. Indeed, since V_1 and V_2 are 1-dimensional, their only subspaces are $\{(0\ 0)\}$ and the whole space. But $U \cap V_1 = \{(0\ 0)\}$ and similarly for V_2, so we cannot write U in the desired form.