Linear Algebra I, Sheet 5, MT2019
Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. Let V, W be vector spaces over $\mathbb{F}$, let $T : V \to W$. Prove that T is linear if and only if $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$. *(This is part of Proposition 28 in the notes.)*

($\Rightarrow$): Suppose that T is linear. Then, by definition, for all $w_1, w_2 \in V$ and $\lambda \in \mathbb{F}$ we have $T(w_1+w_2) = T(w_1) + T(w_2$ and $T(\lambda w_1) = \lambda T(w_1)$.

Take $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$. Then $T(\alpha v_1 + \beta v_2) = T(\alpha v_1) + T(\beta v_2) = \alpha T(v_1) + \beta T(v_2)$.

($\Leftarrow$): Suppose that $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2)$ for all $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$.

Take $w_1, w_2 \in V$ and $\lambda \in \mathbb{F}$.

Then $T(w_1 + w_2) = T(1 \cdot w_1 + 1 \cdot w_2) = T(w_1) + T(w_2)$, and $T(\lambda w_1) = T(\lambda w_1 + 0 \cdot w_2) = \lambda T(w_1) + 0T(w_2) = \lambda T(w_1)$.

S2. Prove Proposition 29 from the notes: for vector spaces V and W over $\mathbb{F}$, the set of linear transformations $V \to W$ forms a vector space (with pointwise addition and scalar multiplication).

We first check closure: take S, $T : V \to W$ are linear maps and $\lambda \in \mathbb{F}$. We need to show that $S + T$ and λS are linear.

Take $v_1, v_2 \in V$ and $\alpha, \beta \in \mathbb{F}$. Then

$$\begin{aligned}
(S+T)(\alpha v_1 + \beta v_2) &= S(\alpha v_1 + \beta v_2) + T(\alpha v_1 + \beta v_2) \text{ by definition of } S+T \\
&= \alpha S(v_1) + \beta S(v_2) + \alpha T(v_1) + \beta T(v_2) \text{ as } S,\, T \text{ linear} \\
&= \alpha(S(v_1) + T(v_1)) + \beta(S(v_2) + T(v_2)) \\
&= \alpha(S+T)(v_1) + \beta(S+T)(v_2) \text{ by definition of } S+T,
\end{aligned}$$

so $S + T$ is linear.

Also,

$$\begin{aligned}
(\lambda S)(\alpha v_1 + \beta v_2) &= \lambda S(\alpha v_1 + \beta v_2) \text{ by definition of } \lambda S \\
&= \lambda(\alpha S(v_1) + \beta S(v_2)) \text{ as } S \text{ linear} \\
&= \alpha(\lambda S)(v_1) + \beta(\lambda S)(v_2)
\end{aligned}$$

so λS is linear.

We then have a list of axioms to check. Most of these are satisfied immediately, because they are inherited from W.

The map sending every vector in V to 0_W is linear, and is the additive identity for addition of linear maps.

For a linear map S, there is a corresponding linear map $-S$, which is its additive inverse.

S3. For each of the following maps, show that it is linear, find the kernel and image, and find the rank and nullity. Check that the Rank-Nullity Theorem is satisfied in each case.

(i) $T_1 : \mathbb{R}^4 \to \mathbb{R}^3$ given by $T(x_1, x_2, x_3, x_4) = (x_2 + x_3 - 2x_4, x_1 - x_2, x_1 - x_3 + 4x_4)$

(ii) $T_2 : \mathbb{R}^3_{\text{col}} \to \mathbb{R}^2_{\text{col}}$ given by $T_2(x) = Ax$ where $A = \begin{pmatrix} 3 & 4 & 7 \\ -1 & 0 & -6 \end{pmatrix}$.

(iii) $T_3 : \mathcal{M}_{m\times n}(\mathbb{R}) \to \mathcal{M}_{n\times m}(\mathbb{R})$ given by $T_3(X) = X^T$.

(i) For $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ and $\alpha, \beta \in \mathbb{R}$ we have

$$\begin{aligned}
&T_1(\alpha(x_1, x_2, x_3, x_4) + \beta(y_1, y_2, y_3, y_4))\\
&\quad = T_1(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, \alpha x_4 + \beta y_4)\\
&\quad = ((\alpha x_2 + \beta y_2) + (\alpha x_3 + \beta y_3) - 2(\alpha x_4 + \beta y_4),\\
&\qquad (\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2), (\alpha x_1 + \beta y_1) - (\alpha x_3 + \beta y_3) + 4(\alpha x_4 + \beta y_4))\\
&\quad = \alpha(x_2 + x_3 - 2x_4, x_1 - x_2, x_1 - x_3 + 4x_4) + \beta(y_2 + y_3 - 2y_4, y_1 - y_2, y_1 - y_3 + 4y_4)\\
&\quad = \alpha T_1(x_1, x_2, x_3, x_4) + \beta T_1(y_1, y_2, y_3, y_4)
\end{aligned}$$

so T_1 is linear.

To find the kernel of T_1, we seek (x_1, x_2, x_3, x_4) such that $x_2 + x_3 - 2x_4 = 0$ and $x_1 - x_2 = 0$ and $x_1 - x_3 + 4x_4 = 0$. We can solve this by a variety of techniques, including interpreting them as a matrix equation

$$\begin{pmatrix} 0 & 1 & 1 & -2 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0.$$

We can use EROs to reduce the matrix to RRE form; we get

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{pmatrix}.$$

We thus read off the solutions, and find that $\ker(T_1) = \operatorname{Span}((-1, -1, 3, 1))$, and so $\operatorname{null}(T_1) = 1$.

To think about the image, we can again use the matrix above. We would want to solve the system of equations where the right-hand side is not just the 0 vector, and so we adjoin a column to the right-hand side of the matrix, to create the corresponding augmented matrix. Looking at the matrix in RRE form, we see that we will be able to sole the resulting system of equations —the key point is that the matrix has row rank 3. So the image of T_1 is $\mathbb{R}^3$, and $\operatorname{rank}(T_1) = 3$.

Rank-Nullity would predict that $4 = \operatorname{rank}(T_1) + \operatorname{null}(T_1)$, and this is indeed true.

(ii) We noted in the lecture notes that any map of this form (premultiplication by a matrix) is linear.

We can use a similar strategy to (i) (note that in fact (i) could be phrased as a map of the same type as (ii), for a suitable choice of matrix). We take matrix A, apply EROs to put it in RRE form, and obtain

$$\begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & -\frac{11}{4} \end{pmatrix}.$$

So we find that the kernel of T_2 is $\mathrm{Span}\left((-6, \frac{11}{4}, 1)\right)$, and $\mathrm{null}(T_2) = 1$.

Again, the map is surjective, with image $\mathbb{R}^2_{\mathrm{col}}$ and $\mathrm{rank}(T_2) = 2$.

This time, Rank-Nullity predicts that $3 = \mathrm{rank}(T_2) + \mathrm{null}(T_2)$, which is true.

(iii) We checked that this map is linear in Sheet 1 Q5.

We see that X is in the kernel of T_3 if and only if $X^T = 0_{n\times m}$ if and only if $X = 0_{m\times n}$. So the kernel of T_3 is $\{0_{m\times n}\}$ and $\mathrm{null}(T_3) = 0$.

Also, T_3 is clearly surjective because for any $Y \in \mathcal{M}_{n\times m}(\mathbb{R})$ we have $T_3(Y^T) = Y$. So the image of T_3 is $\mathcal{M}_{n\times m}(\mathbb{R})$, with $\mathrm{rank}(T_3) = mn$.

Now Rank-Nullity predicts that $mn = \mathrm{rank}(T_3) + \mathrm{null}(T_3)$, which holds.