Linear Algebra I, Sheet 5, MT2019

Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

Vicky Neale (vicky.neale@maths)

P1. Consider the usual dot product on $\mathbb{R}^3$: we define $(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1y_1 + x_2y_2 + x_3y_3$. Fix a vector $(a_1, a_2, a_3) \in \mathbb{R}^3$, and define a map $T : \mathbb{R}^3 \to \mathbb{R}$ by $T(x_1, x_2, x_3) = (x_1, x_2, x_3) \cdot (a_1, a_2, a_3)$. Show that T is linear. Find its kernel and image, and its rank and nullity.

<u>Claim</u> T is linear.
<u>Proof</u> Take $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$ and $\lambda, \mu \in \mathbb{R}$. Then

$$\begin{aligned} T(\lambda(x_1, x_2, x_3) + \mu(y_1, y_2, y_3)) &= T(\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \lambda x_3 + \mu y_3) \\ &= (\lambda x_1 + \mu y_1, \lambda x_2 + \mu y_2, \lambda x_3 + \mu y_3) \cdot (a_1, a_2, a_3) \\ &= (\lambda x_1 + \mu y_1)a_1 + (\lambda x_2 + \mu y_2)a_2 + (\lambda x_3 + \mu y_3)a_3 \\ &= \lambda(x_1a_1 + x_2a_2 + x_3a_3) + \mu(y_1a_1 + y_2a_2 + y_3a_3) \\ &= \lambda T(x_1, x_2, x_3) + \mu T(y_1, y_2, y_3). \end{aligned}$$

So T is linear. □

Now $(x_1, x_2, x_3) \in \ker T$ if and only if $(x_1, x_2, x_3) \cdot (a_1, a_2, a_3) = 0$, that is, if and only if (x_1, x_2, x_3) lies in the plane through the origin with normal vector (a_1, a_2, a_3), so the kernel of T is this plane. The exception here is if $(a_1, a_2, a_3) = 0$, when the kernel of T is the whole of $\mathbb{R}^3$.

If $(a_1, a_2, a_3) = 0$ then the image of T is just $\{0\}$. Otherwise, the image of T is the whole of $\mathbb{R}$ — it is a subspace of $\mathbb{R}$ and is not just the 0 vector.

So we see that if $(a_1, a_2, a_3) = 0$ then $\text{null}(T) = 3$ and $\text{rank}(T) = 0$, and otherwise we have $\text{null}(T) = 2$ and $\text{rank}(T) = 1$.

Reassuringly, in either case we have $\text{rank}(T) + \text{null}(T) = 3$, *as expected.*

This linear map will reappear in the final section of the course, when we explore bilinear forms and inner products.

P2. Let $T : \mathbb{R}^7 \to \mathbb{R}^4$ be a linear map. What are the possible nullities of T? Justify your answer. For each such possibility, give an example of a linear map $T : \mathbb{R}^7 \to \mathbb{R}^4$ with that nullity.

By Rank-Nullity, we have $\operatorname{rank}(T) + \operatorname{null}(T) = 7$.

Also, the image of T is a subspace of $\mathbb{R}^4$, so $0 \leqslant \operatorname{rank}(T) \leqslant 4$.

Combining these, we see that $3 \leqslant \operatorname{null}(T) \leqslant 7$. This eliminates many nonnegative integers immediately; we have 5 cases remaining.

Our plan is to show that in each case it is *possible to have such a linear map, by exhibiting one. In each case we will endeavour to find an example where it is relatively easy to see that it has the required properties.*

Consider $T_3 : \mathbb{R}^7 \to \mathbb{R}^4$ given by $T_3(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (x_1, x_2, x_3, x_4)$. This is linear, and clearly has kernel $\{(0, 0, 0, 0, x_5, x_6, x_7) : x_5, x_6, x_7 \in \mathbb{R}\}$ so $\operatorname{null}(T_3) = 3$. Also, we see immediately that the image of T_3 is $\mathbb{R}^4$, so $\operatorname{rank}(T_3) = 4$ as expected.

Consider $T_4 : \mathbb{R}^7 \to \mathbb{R}^4$ given by $T_4(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (x_1, x_2, x_3, 0)$. This is linear. We see that it has kernel $\{(0, 0, 0, x_4, x_5, x_6, x_7) : x_4, x_5, x_6, x_7 \in \mathbb{R}\}$ so $\operatorname{null}(T_4) = 4$, and $\operatorname{rank}(T_4) = 3$.

Consider $T_5 : \mathbb{R}^7 \to \mathbb{R}^4$ given by $T_5(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (x_1, x_2, 0, 0)$. This is linear, with $\operatorname{null}(T_5) = 5$ and $\operatorname{rank}(T_5) = 2$.

Consider $T_6 : \mathbb{R}^7 \to \mathbb{R}^4$ given by $T_6(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (x_1, 0, 0, 0)$. This is linear, with $\operatorname{null}(T_6) = 6$ and $\operatorname{rank}(T_6) = 1$.

Finally, consider $T_7 : \mathbb{R}^7 \to \mathbb{R}^4$ given by $T_7(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (0, 0, 0, 0)$. This is linear, with $\operatorname{null}(T_7) = 7$ and $\operatorname{rank}(T_7) = 0$.

So we see that n can be the nullity of such a linear map if and only if $3 \leqslant n \leqslant 7$.

P3. Consider the real vector space V consisting of all real (infinite) sequences. Consider the map $T : V \to V$ given by $T_1(x_1, x_2, x_3, \dots) = (0, x_1, x_2, x_3, \dots)$. Show that T_1 is linear. What is the kernel of T_1? What is its image?
Define T_2 by $T_2(x_1, x_2, x_3, \dots) = (x_1 + x_2, 0, x_3, x_4, 0, 0, 0, \dots)$. Show that T_2 is linear. What is the kernel of T_2? What is its image?
Can you give an example of a linear map $T_3 : V \to V$ where neither the kernel nor the image of T_3 has finite dimension?

We can check that T_1 and T_2 are linear in the usual way; we do not give the details here.

We see that $(x_1, x_2, x_3, \dots) \in \ker T_1$ if and only if $(0, x_1, x_2, x_3, \dots) = (0, 0, 0, 0, \dots)$ if and only if $x_1 = x_2 = x_3 = \cdots = 0$. So $\ker(T_1) = \{(0, 0, 0, \dots)\}$.

We find that the image of T_1 is the space of all real sequences with first term 0.

Now $(x_1, x_2, x_3, \dots) \in \ker T_2$ if and only if $(x_1 + x_2, 0, x_3, x_4, 0, 0, 0, \dots) = (0, 0, 0, 0, 0, \dots)$ if and only if $x_2 = -x_1$ and $x_3 = x_4 = 0$. So $\ker T_2 = \{(x_1, -x_1, 0, 0, x_5, x_6, \dots) : x_1, x_5, x_6, \dots \in \mathbb{R}\}$.

Finally, the image of T_2 is $\{(a, 0, b, c, 0, 0, 0, \dots) : a, b, c \in \mathbb{R}\}$.

Consider for example $T_3(x_1, x_2, x_3, x_4, \dots) = (x_1, 0, x_3, 0, x_5, 0, \dots)$. This is linear, and we see that neither the image nor the kernel has a finite basis.