Linear Algebra I, Sheet 6, MT2019

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. In each part below, you are given two vector spaces V and W, a linear map $T : V \to W$, and ordered bases B_V and B_W for V and W respectively. Find the matrix A of T with respect to these bases.

(i) $V = W = \mathbb{R}^3$, with standard basis $B_V = B_W$, and $T(x, y, z) = (y, z, 0)$.

(ii) $V = \mathbb{R}^5$ with standard basis B_V, and $W = \mathbb{R}^3$ with ordered basis $w_1 = (-1, 1, 0)$, $w_2 = (1, 0, 1)$, $w_3 = (0, 2, 1)$, and $T(x_1, x_2, x_3, x_4, x_5) = (x_1 - x_3 + x_4, x_1 + x_2, x_5)$.

(iii) $V = W = \mathbb{R}^3$, with ordered bases B_V given by $v_1 = (3, -2, 0)$, $v_2 = (1, 1, 1)$, $v_3 = (4, 7, -2)$ and B_W given by $w_1 = (1, 0, 0)$, $w_2 = (0, 1, 1)$, $w_3 = (0, 1, -1)$, and T the identity map.

To find the matrix A for T, we apply T to each basis vector in B_V, write it in terms of B_W, and use these to form the columns of A.

(i) We compute

$$\begin{aligned} T(1,0,0) &= (0,0,0) \\ T(0,1,0) &= (1,0,0) \\ T(0,0,1) &= (0,1,0) \end{aligned}$$

so find that

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(ii) We compute

$$\begin{aligned} T(1,0,0,0,0) &= (1,1,0) = -3(-1,1,0) - 2(1,0,1) + 2(0,2,1) = -3w_1 - 2w_2 + 2w_3 \\ T(0,1,0,0,0) &= (0,1,0) = -(-1,1,0) - (1,0,1) + (0,2,1) = -w_1 - w_2 + w_3 \\ T(0,0,1,0,0) &= (-1,0,0) = 2(-1,1,0) + (1,0,1) - (0,2,1) = 2w_1 + w_2 - w_3 \\ T(0,0,0,1,0) &= (1,0,0) = -2w_1 - w_2 + w_3 \\ T(0,0,0,0,1) &= (0,0,1) = 2(-1,1,0) + 2(1,0,1) - (0,2,1) = 2w_1 + 2w_2 - w_3 \end{aligned}$$

so find that

$$A = \begin{pmatrix} -3 & -1 & 2 & -2 & 2 \\ -2 & -1 & 1 & -1 & 2 \\ 2 & 1 & -1 & 1 & -1 \end{pmatrix}.$$

(iii) We compute

$$\begin{aligned}
T(v_1) &= T(3,-2,0) = (3,-2,0) = 3(1,0,0) - (0,1,1) - (0,1,-1) = 3w_1 - w_2 - w_3 \\
T(v_2) &= T(1,1,1) = (1,1,1) = (1,0,0) + (0,1,1) = w_1 + w_2 \\
T(v_3) &= T(4,7,-2) = (4,7,-2) = 4(1,0,0) + \frac{5}{2}(0,1,1) + \frac{9}{2}(0,1,-1) = 4w_1 + \frac{5}{2}w_2 + \frac{9}{2}w_3
\end{aligned}$$

so find that

$$A = \begin{pmatrix} 3 & 1 & 4 \\ -1 & 1 & \frac{5}{2} \\ -1 & 0 & \frac{9}{2} \end{pmatrix}.$$

S2. Let $V = \mathbb{R}_3[x]$ be the real vector space of polynomials of degree at most 3 (as in Sheet 5 Q3). Let B_3 be the ordered basis x^3, x^2, x, 1. For each of the following linear maps $T : V \to V$, find the matrix for T with respect to B_3.

(i) $T = D$, differentiation.

(ii) $T(p(x)) = p(x+1)$.

(iii) $T(p(x)) = \int_0^1 (t-x)^3 p(t)dt$

In each case, we apply T to each basis vector in turn, and the coefficients with respect to the basis form the columns of the matrix.

(i) We compute

$$\begin{aligned}
T(x^3) &= 3x^2 \\
T(x^2) &= 2x \\
T(x) &= 1 \\
T(1) &= 0
\end{aligned}$$

and so the matrix is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(ii) We compute

$$\begin{aligned}
T(x^3) &= (x+1)^3 = x^3 + 3x^2 + 3x + 1 \\
T(x^2) &= (x+1)^2 = x^2 + 2x + 1 \\
T(x) &= x + 1 \\
T(1) &= 1
\end{aligned}$$

and so the matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(iii) For $0 \leqslant i \leqslant 3$, we have

$$\begin{aligned}
T(x^i) &= \int_0^1 (t-x)^3 t^i dt \\
&= \int_0^1 t^{3+i} - 3xt^{2+i} + 3x^2t^{1+i} - x^3t^i dt \\
&= \left[\frac{1}{4+i}t^{4+i} - \frac{3}{3+i}xt^{3+i} + \frac{3}{2+i}x^2t^{2+i} - \frac{1}{i+1}x^3t^{i+1}\right]_0^1 \\
&= \frac{1}{4+i} - \frac{3}{3+i}x + \frac{3}{2+i}x^2 - \frac{1}{1+i}x^3,
\end{aligned}$$

and so the matrix is

$$\begin{pmatrix} -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & -1 \\ \frac{3}{5} & \frac{3}{4} & 1 & \frac{3}{2} \\ -\frac{1}{2} & -\frac{3}{5} & -\frac{3}{4} & -1 \\ \frac{1}{7} & \frac{1}{6} & \frac{1}{5} & \frac{1}{4} \end{pmatrix}.$$

S3. For each of the following matrices, use elementary row operations and elementary column operations to reduce the matrix to a block matrix with an identity matrix in the top left-hand corner and 0 elsewhere (as in Proposition 47 of the notes).

(a) $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 7 & 4 \\ 5 & -1 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 4 & 9 \\ -2 & 8 & -7 & 3 \\ 1 & 5 & -6 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 7 & -1 & 2 \\ 1 & 0 & -3 \\ 5 & -1 & 8 \end{pmatrix}$

We use elementary row operations to take the matrix to reduced row echelon form, then elementary column operations to finish off.

(a) Applying EROs takes the given matrix to I_3 in RRE form, and so there is no need to apply ECOs: we can stop with I_3.

(b) Applying EROs takes the given matrix to the following matrix in RRE form:

$$\begin{pmatrix} 1 & 0 & 0 & \frac{3}{7} \\ 0 & 1 & 0 & \frac{183}{91} \\ 0 & 0 & 1 & \frac{159}{91} \end{pmatrix}.$$

Now we apply ECOs: we subtract suitable multiples of columns 1, 2, 3 to clear column 4, and finish with matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(c) We apply EROs, and find that the given matrix reduces to

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -23 \\ 0 & 0 & 0 \end{pmatrix}$$

in RRE form. Now we apply ECOs: we subtract suitable multiples of columns 1 and 2 to clear column 3, and finish with matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$