Linear Algebra I, Sheet 6, MT2019

Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

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P1. S2 asks you to find the matrix A for differentiation of polynomials of degree at most 3, with respect to the ordered basis x^3, x^2, x, 1. Now find the matrix B with respect to the ordered basis 1, x, x^2, x^3. Can you find a matrix P such that $B = P^{-1}AP$?

As in S2, the matrix A is

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

We have options about how to tackle this question. We could compute the matrix directly, by applying the linear map of differentiation to each basis vector in turn, to get the entries in the columns of the matrix. Having done that work in S2, we can write down the matrix immediately, it's

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We could also use the Change of Basis Theorem. We find that the change of basis matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

and this is self-inverse, so the Change of Basis Theorem tells us that

$$\begin{aligned} B &= P^{-1}AP \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

as expected.

P2. Is it true that if A, B and C are $n \times n$ real matrices then $\mathrm{tr}(ABC) = \mathrm{tr}(BAC)$? Is it true that $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$? (Give a proof or counterexample for each.)

(i) This is not true. For example, let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned} ABC &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} BAC &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 \end{aligned}$$

so $\mathrm{tr}(ABC) = 1$ but $\mathrm{tr}(BAC) = 0$.

(ii) This is true. Let $X = A$ and $Y = BC$. Then as in Q4(a) on the sheet, $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ so $\mathrm{tr}(ABC) = \mathrm{tr}(BCA)$.

P3. Let A be an $n \times n$ real matrix. For each of the following, give a proof or a counterexample.

(i) If A is similar to 0, then $A = 0$.

(ii) If A is similar to I_n, then $A = I_n$.

(iii) If A is similar to a diagonal matrix, then A is diagonal.

(iv) If A is similar to a symmetric matrix, then A is symmetric.

(v) If A is similar to B, then A^2 is similar to B^2.

(i) This is true. If A is similar to 0, then there is an invertible matrix P such that $A = P^{-1}0P = 0$.

(ii) This is true. If A is similar to I_n, then there is an invertible matrix P such that $A = P^{-1}I_nP = I_n$.

(iii) This is not true. For example, the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$$

is similar to the diagonal matrix $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ but is not diagonal.

(iv) This is not true. We can take exactly the same counterexample as in (iii), because a diagonal matrix is symmetric.

(v) This is true. If A is similar to B, then there is an invertible matrix P such that $A = P^{-1}BP$. Then $A^2 = (P^{-1}BP)(P^{-1}BP) = P^{-1}B^2P$, so A^2 is similar to B^2.