Linear Algebra I, Sheet 7, MT2019

Starter

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

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S1. Let $V = \mathcal{M}_{2\times 2}(\mathbb{R})$. For A, $B \in \mathcal{M}_{2\times 2}(\mathbb{R})$, define $\langle A, B\rangle = \mathrm{tr}(A^T B)$. Does this define an inner product on V?

We need to check bilinearity, symmetry, and positive definiteness.

<u>bilinearity</u> For A_1, A_2 and B in $\mathcal{M}_{2\times 2}(\mathbb{R})$ and λ, $\mu \in \mathbb{R}$, we have

$$\begin{aligned}\langle \lambda A_1 + \mu A_2, B\rangle &= \mathrm{tr}((\lambda A_1 + \mu A_2)^T B) \\ &= \mathrm{tr}((\lambda A_1^T + \mu A_2^T)B) \\ &= \mathrm{tr}(\lambda A_1^T B + \mu A_2^T B) \\ &= \lambda\,\mathrm{tr}(A_1^T B) + \mu\,\mathrm{tr}(A_2^T B) \\ &= \lambda\langle A_1, B\rangle + \mu\langle A_2, B\rangle\end{aligned}$$

as required, using properties of trace.

We can do a similar thing in the second argument, or use symmetry (which we are about to prove).

<u>symmetry</u> For A, $B \in \mathcal{M}_{2\times 2}(\mathbb{R})$, we have

$$\langle A, B\rangle = \mathrm{tr}(A^T B) = \mathrm{tr}((A^T B)^T) = \mathrm{tr}(B^T A) = \langle B, A\rangle.$$

<u>positive definite</u> For $A \in \mathcal{M}_{2\times 2}(\mathbb{R}$, say $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$\begin{aligned}\langle A, A\rangle &= \mathrm{tr}(A^T A) \\ &= \mathrm{tr}\begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} \\ &= a^2 + c^2 + b^2 + d^2 \\ &\geqslant 0\end{aligned}$$

with equality if and only if $a = b = c = d = 0$, that is, if and only if $A = 0$.

So this is indeed an inner product.

S2. For each given inner product space V and vector $u \in V$, find a basis for the space $u^\perp$.

(i) $V = \mathbb{R}^3$ with the usual dot product, $u = (3, -1, 2)$.

(ii) $V = \mathbb{R}_2[x]$ with inner product as in Q3(b) overleaf, $u(x) = x^2 - 5x + 6$.

(i) By definition, here $u^\perp = \{(x, y, z) : 3x - y + 2z = 0\}$. We see that $u^\perp$ contains $(2, 0, -3)$ and $(0, 2, 1)$, and these are clearly linearly independent. In addition, they span $u^\perp$, because if $(x, y, z) \in u^\perp$ then $y = 3x + 2z$ and so $(x, y, z) = \frac{x}{2}(2, 0, -3) + (z + \frac{3x}{2})(0, 2, 1)$.

So $\{(2, 0, -3), (0, 2, 1)\}$ is a basis for $u^\perp$.

We could also have used the fact from lectures that $\dim(u^\perp) = \dim V - 1$, *so once we had a linearly independent set of 2 vectors in* $u^\perp$ *it had to be a basis.*

(ii) Let $f(x) = a_2x^2 + a_1x + a_0 \in \mathbb{R}_2[x]$ be in $u^\perp$. Then $\langle f, u\rangle = 0$, that is, $f(0)u(0) + f(1)u(1) + f(2)u(2) = 0$.

For this choice of f and u, this becomes $6a_0 + 2(a_2 + a_1 + a_0) = 0$, that is, $a_2 + a_1 + 4a_0 = 0$. So we see that $f(x) = (-a_1 - 4a_0)x^2 + a_1x + a_0$ for some a_2, a_1, $a_0 \in \mathbb{R}$. A quick check confirms that all such functions are in $u^\perp$.

So to find a basis, we might choose $f_1(x) = -4x^2 + 1$ and $f_2(x) = -x^2 + x$. These are two linearly independent vectors in a 2-dimensional space (note that $\dim V = 3$ so $\dim u^\perp = 2$), so form a basis.

S3. For $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in \mathbb{C}^n$, define $\langle (x_1, \dots, x_n), (y_1, \dots, y_n)\rangle = x_1y_1 + \cdots + x_ny_n$. Does this define a Hermitian form on $\mathbb{C}^n$? (Compare with Q6(b).)

No, it does not. It is not sesquilinear. For example,

$$\langle (1, 0, \dots, 0), (i, 0, \dots, 0)\rangle = i$$

but

$$\overline{\langle (i, 0, \dots, 0), (1, 0, \dots, 0)\rangle} = -i.$$