Linear Algebra I, Sheet 7, MT2019

Pudding

I would really appreciate feedback on ways in which these comments and solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

I'm not going to give full details/proofs for every question, but hopefully I'll give something useful against which you can compare your thinking.

Vicky Neale (vicky.neale@maths)

P1. Let V be a real inner product space. Suppose that v_1, $v_2 \in V$ have the property that $\langle v_1, v\rangle = \langle v_2, v\rangle$ for all $v \in V$. Does this mean that $v_1 = v_2$? Give a proof or counterexample.

Suppose that $\langle v_1, v\rangle = \langle v_2, v\rangle$ for all $v \in V$.
Then, using bilinearity, we have $\langle v_1 - v_2, v\rangle = 0$ for all $v \in V$.
In particular, this holds when $v = v_1 - v_2$, so $\langle v_1 - v_2, v_1 - v_2\rangle = 0$.
But the inner product is positive definite, so $v_1 - v_2 = 0$.

P2. Let $V = \mathbb{R}_2[x]$ with the inner product defined in Q3(b). Find an orthonormal basis of V.

Note that V is 3-dimensional (for example, 1, x, x^2 is a basis). So it is enough to find a set of three orthonormal vectors — these must be linearly independent and hence a basis.

Sadly 1, x, x^2 are *not* orthonormal. But we can try to adapt them.

We see that $\langle 1, 1\rangle = 3$. So let's scale, and replace 1 by $\frac{1}{\sqrt{3}}$, because $\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\rangle = 1$.

Now $\langle \frac{1}{\sqrt{3}}, x\rangle = \frac{1}{\sqrt{3}} + 2\frac{1}{\sqrt{3}} = \sqrt{3}$ — not, as we would hope, 0. But if we subtract off a suitable multiple of $\frac{1}{\sqrt{3}}$, then we can fix things. Specifically, instead of x, let's consider $x - \sqrt{3}\frac{1}{\sqrt{3}} = x - 1$.

To reassure ourselves, we can check: we have $\langle \frac{1}{\sqrt{3}}, x-1\rangle = -\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = 0$, so $\frac{1}{\sqrt{3}}$ and $x - 1$ are indeed orthogonal.

Now $\langle x-1, x-1\rangle = 1 + 1 = 2$, so we can rescale — we should consider $\frac{1}{\sqrt{2}}(x-1)$ instead, as this has length 1. Note that rescaling doesn't affect it being orthogonal to $\frac{1}{\sqrt{3}}$.

So far we have included $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{2}}(x-1)$ in our efforts towards an orthonormal basis. Let's try a similar strategy to adapt x^2.

We have $\langle \frac{1}{\sqrt{3}}, x^2\rangle = \frac{1}{\sqrt{3}} + 4\frac{1}{\sqrt{3}} = 5\frac{1}{\sqrt{3}}$ and $\langle \frac{1}{\sqrt{2}}(x-1), x^2\rangle = 4\frac{1}{\sqrt{2}} = 2\sqrt{2}$. So if we consider not x^2 but instead $x^2 - 5\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} - 2\sqrt{2}\frac{1}{\sqrt{2}}(x-1) = x^2 - 2x + \frac{1}{3}$, then we'll have a vector orthogonal to $\frac{1}{\sqrt{3}}$ and $\frac{1}{\sqrt{2}}(x-1)$.

Now $\langle x^2 - 2x + \frac{1}{3}, x^2 - 2x + \frac{1}{3}\rangle = \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{2}{3}$, so let's rescale to consider instead $\frac{\sqrt{3}}{\sqrt{2}}(x^2 - 2x + \frac{1}{3})$, which has length 1 and is still orthogonal to the other two vectors.

So we see that

$$\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}(x-1), \frac{\sqrt{3}}{\sqrt{2}}(x^2 - 2x + \frac{1}{3})\right\}$$

is an orthonormal basis.

There are lots of possible approaches that we might use to find an orthonormal basis. The approach I've chosen here is a strategy that works in general, called the Gram-Schmidt procedure. You will learn about this in Linear Algebra II next term.

P3. Let V be a real vector space with inner product $\langle -, - \rangle$ and associated length function $\| \cdot \|$. Show that $\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 = 2\|v_1\|^2 + 2\|v_2\|^2$ for all v_1, $v_2 \in V$ — this is called the *parallelogram law*.

Take v_1, $v_2 \in V$.
Then, using bilinearity and symmetry of the inner product,

$$\begin{aligned}
\|v_1 + v_2\|^2 + \|v_1 - v_2\|^2 &= \langle v_1 + v_2, v_1 + v_2 \rangle + \langle v_1 - v_2, v_1 - v_2 \rangle \\
&= \langle v_1, v_1 \rangle + 2\langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle + \langle v_1, v_1 \rangle - 2\langle v_1, v_2 \rangle + \langle v_2, v_2 \rangle \\
&= 2\langle v_1, v_1 \rangle + 2\langle v_2, v_2 \rangle \\
&= 2\|v_1\|^2 + 2\|v_2\|^2.
\end{aligned}$$