Complex Numbers Problems Sheet, MT2019

I would really appreciate feedback on ways in which these solutions could be improved and made more helpful, so please let me know about typos (however trivial), mistakes, alternative solutions, or additional comments that might be useful.

Vicky Neale (vicky.neale@maths)

1. Which of the following quadratic equations have real roots, which do not?

$$3x^2 + 2x - 1 = 0; \quad 2x^2 - 6x + 9 = 0; \quad -4x^2 + 7x - 9 = 0; \quad 4x^2 - 9x + 5 = 0.$$

(i) The discriminant is $2^2 - 4 \cdot 3 \cdot (-1) = 4 + 12 > 0$, so the equation has two distinct real roots.

(ii) The discriminant is $(-6)^2 - 4 \cdot 2 \cdot 9 = 36 - 72 < 0$, so the equation has no real roots.

(iii) The discriminant is $7^2 - 4 \cdot (-4) \cdot (-9) = 49 - 144 < 0$, so the equation has no real roots.

(iv) The discriminant is $(-9)^2 - 4 \cdot 4 \cdot 5 = 81 - 80 > 0$, so the equation has two distinct real roots.

2. Write a careful proof of the theorem that if $z, w \in \mathbb{C}$ then $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.

Claim Take $z, w \in \mathbb{C}$. Then $\overline{z+w} = \overline{z} + \overline{w}$.
Proof We may write $z = a + bi$ and $w = c + di$, where $a, b, c, d \in \mathbb{R}$. Then

$$\begin{aligned} \overline{z+w} &= \overline{(a+bi)+(c+di)} \\ &= \overline{(a+c)+(b+d)i} \\ &= (a+c)-(b+d)i \\ &= (a-bi)+(c-di) \\ &= \overline{z} + \overline{w}. \end{aligned}$$

□

Claim Take $z, w \in \mathbb{C}$. Then $\overline{zw} = \overline{z}\,\overline{w}$.
Proof Write $z = a + bi$ and $w = c + di$ as above. Then

$$\begin{aligned} \overline{zw} &= \overline{(a+bi)(c+di)} \\ &= \overline{(ac-bd)+(ad+bc)i} \\ &= (ac-bd)-(ad+bc)i \\ &= (a-bi)(c-di) \\ &= \overline{z}\,\overline{w}. \end{aligned}$$

□

3. Put each of the following complex numbers into standard form $a + bi$:

$$(1+2i)(3-i); \quad (2+i)(1-2i); \quad (1+i)^4; \quad (1-\sqrt{3}i)^3; \quad \frac{7-2i}{5+12i}; \quad \frac{i}{1-i}.$$

(i) We have

$$\begin{aligned}(1+2i)(3-i) &= 3+6i-i-2i^2 \\ &= 3+5i+2 \\ &= 5+5i.\end{aligned}$$

[That is not what I would actually write for my own solution. I have had enough practice that I don't write i^2, I just replace it by -1 straight away. And in fact I have had so much practice that I would do the calculations in my head and just write

$$(1+2i)(3-i) = 5+5i.$$

I wrote out the detailed steps above to show what I'm doing in my head, but I'm not going to write it all out again in these solutions. And far from encouraging you to write out all the steps, I'd encourage you to develop your fluency to the point where you can just write down the answer to things like this. Sometimes it's actively helpful to write down intermediate steps, but this seems to me like one of those occasions when it just gets in the way.]

(ii) $(2+i)(1-2i) = 4-3i$

(iii) $(1+i)^4 = (2i)^2 = -4$

That's intriguing! You might like to think about how to interpret this geometrically, if you haven't already done so.

(iv) $(1-\sqrt{3}i)^3 = (-2-2\sqrt{3}i)(1-\sqrt{3}i) = -8$

Again, how could we see this geometrically?

(v)

$$\frac{7-2i}{5+12i} = \frac{(7-2i)(5-12i)}{(5+12i)(5-12i)} = \frac{11}{169} - \frac{94}{169}i$$

(vi)

$$\frac{i}{1-i} = \frac{i(1+i)}{(1-i)(1+i)} = -\frac{1}{2} + \frac{1}{2}i$$

4. Find the modulus and argument of each of the following complex numbers:

$$1+\sqrt{3}i; \quad (2+i)(3-i); \quad (1+i)^5.$$

(i) We have

$$|1+\sqrt{3}i| = \sqrt{1^2+(\sqrt{3})^2} = \sqrt{4} = 2$$

and

$$\arg(1+\sqrt{3}i) = \arctan(\sqrt{3}) = \frac{\pi}{3}$$

Here, and elsewhere in the question, we give one value for the argument, but adding $2k\pi$ for any integer k would also give another possible value.

(ii) We have $(2+i)(3-i) = 7+i$, so

$$|(2+i)(3-i)| = \sqrt{7^2+1^2} = \sqrt{50} = 5\sqrt{2}$$

and

$$\arg((2+i)(3-i)) = \arctan\left(\frac{1}{7}\right)$$

(iii) We have

$$|(1+i)^5| = |1+i|^5 = 4\sqrt{2}$$

and

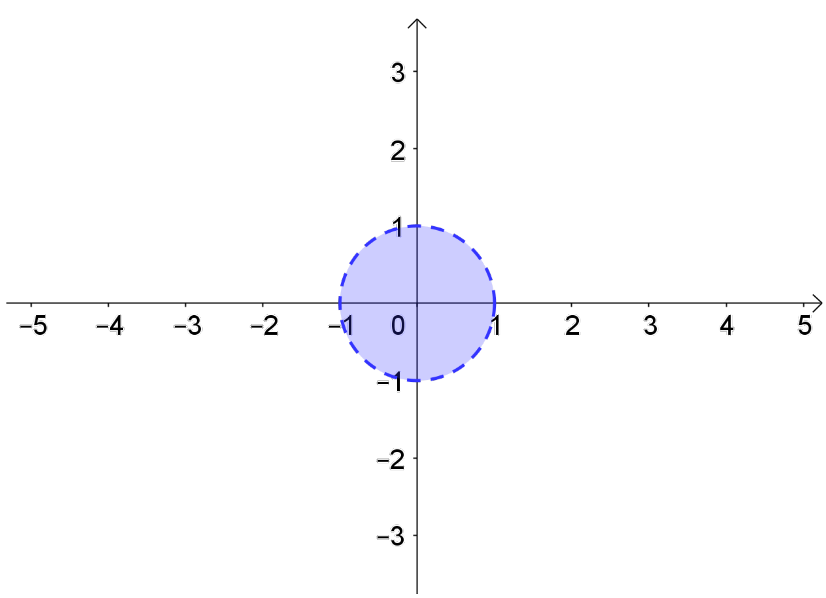
$$\arg((1+i)^5) = 5\arg(1+i) = \frac{5\pi}{4}.$$

Of course we could have expanded out $(1+i)^5$ in the form $a+bi$ and then found the modulus and argument, but I think that using properties of modulus and argument saved us some work.

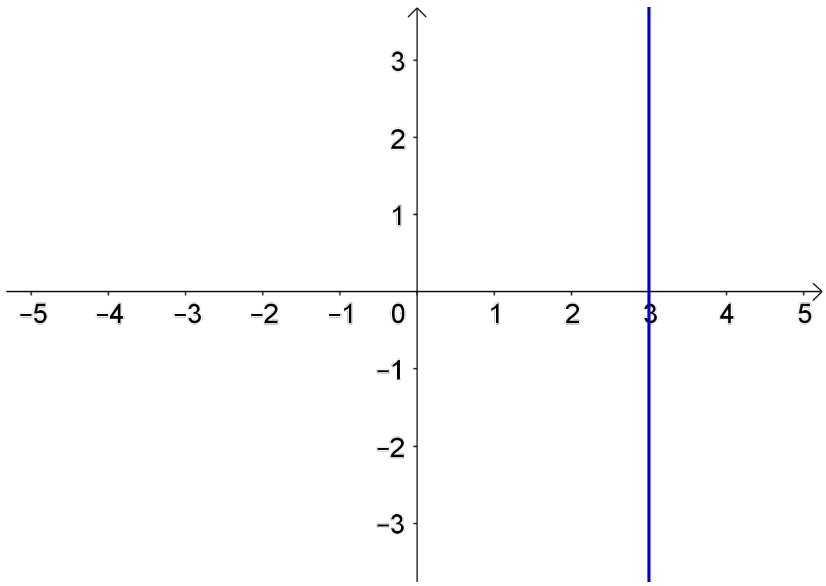
5. On separate Argand diagrams sketch each of the following subsets of $\mathbb{C}$:

$A := \{z : |z| < 1\}$; $\quad B := \{z : \operatorname{Re} z = 3\}$; $\quad C := \{z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$;
$D := \{z : \arg(z-i) = \frac{\pi}{2}\}$; $\quad E := \{z : |z-3-4i| = 5\}$; $\quad F := \{z : |z-1| = |z-i|\}$.

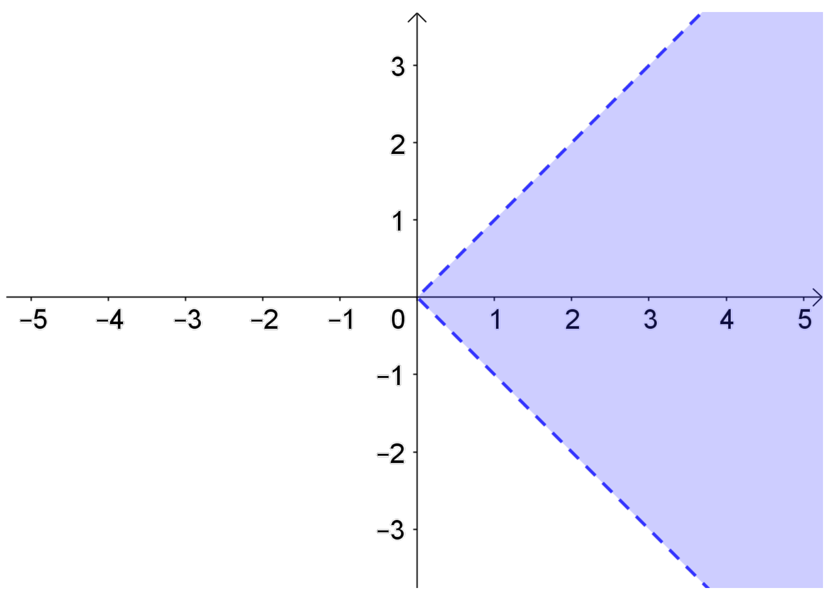
$A := \{z : |z| < 1\}$ This is the interior of a circle of radius 1 centred on the origin (the boundary circle is not included).



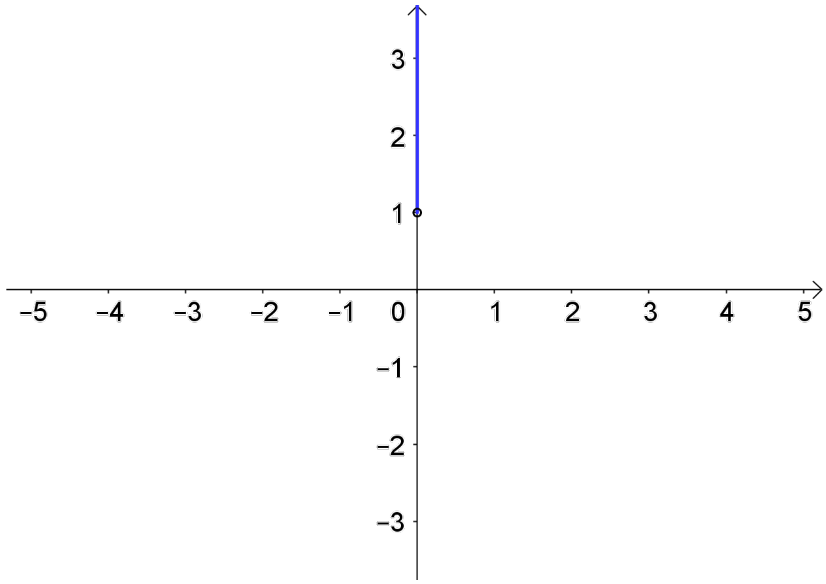
$B := \{z : \mathrm{Re}\, z = 3\}$ This is a vertical line through the point 3 on the real axis.



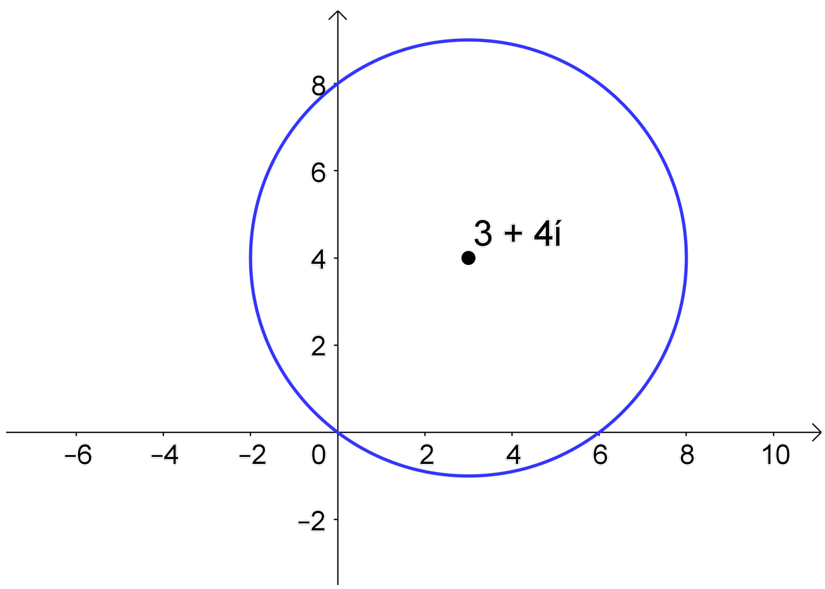
$C := \{z : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$ This is a quarter sector, not including the boundary half-lines or the origin. The boundary half lines are rays from the origin with gradient 1 and -1 respectively, and to the right of the vertical axis.



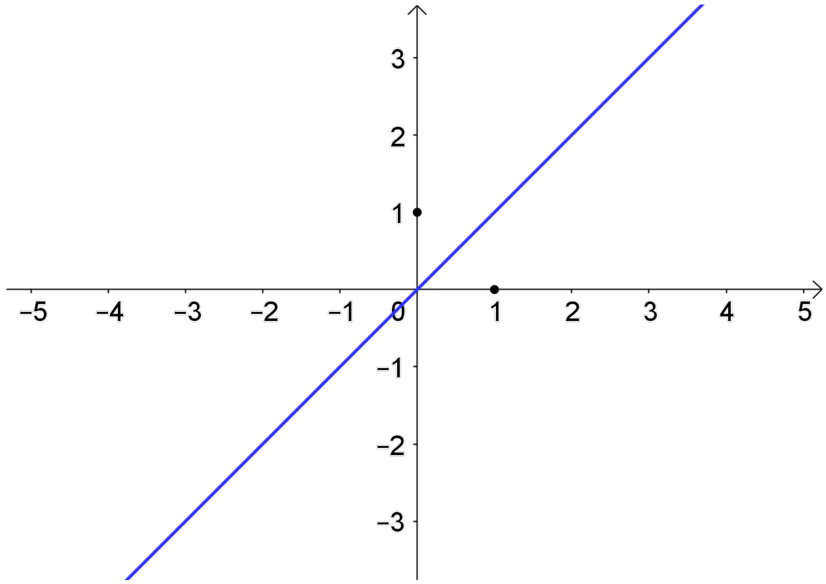
$D := \{z : \arg(z - i) = \frac{\pi}{2}\}$ This is a vertical half-line from i upwards, not including i.



$E := \{z : |z - 3 - 4i| = 5\}$ This is a circle of radius 5 with centre $3 + 4i$.



$F := \{z : |z-1| = |z-i|\}$ This is the set of points at equal distance from 1 and i, which gives a straight line through the origin with gradient 1 (putting that another way, with equation $y = x$).



6. For each of the following complex numbers w, what transformation of the Argand diagram does multiplication by w represent?

$$i; \quad (1+i); \quad (1-i); \quad (3+4i).$$

(i) Multiplication by i corresponds to a rotation by $\frac{\pi}{2}$ anticlockwise about the origin.

(ii) Multiplication by $1+i$ corresponds to an enlargement by factor $\sqrt{2}$ from the origin and a rotation by $\frac{\pi}{4}$ anticlockwise about the origin.

(iii) Multiplication by $1-i$ corresponds to an enlargement by factor $\sqrt{2}$ from the origin and a rotation by $\frac{\pi}{4}$ clockwise about the origin.

(iv) Multiplication by $3+4i$ corresponds to an enlargement by factor 5 from the origin and a rotation by $\arctan\frac{4}{3}$ anticlockwise about the origin.

7. Use De Moivre's Theorem to show that if $\theta \in \mathbb{R}$ then

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta; \quad \sin 5\theta = (16\cos^4\theta - 12\cos^2\theta + 1)\sin\theta.$$

<u>Claim</u> Take $\theta \in \mathbb{R}$. Then

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta; \quad \sin 5\theta = (16\cos^4\theta - 12\cos^2\theta + 1)\sin\theta.$$

<u>Proof</u> By de Moivre's theorem we have

$$(\cos\theta + i\sin\theta)^5 = \cos(5\theta) + i\sin(5\theta).$$

Expanding out the left-hand side, we obtain

$$(\cos\theta + i\sin\theta)^5 = (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta),$$

and comparing real and imaginary parts with $\cos(5\theta) + i\sin(5\theta)$ we obtain

$$\begin{aligned}
\cos(5\theta) &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
&= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\
&= \cos^5\theta - 10\cos^2\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta
\end{aligned}$$

and

$$\begin{aligned}
\sin(5\theta) &= 5\cos^5\theta\sin\theta - 10\cos^2\theta\sin^2\theta + \sin^5\theta \\
&= \sin\theta(5\cos^4\theta - 10\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2) \\
&= \sin\theta(5\cos^4\theta - 10\cos^2\theta + 10\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta) \\
&= \sin\theta(16\cos^4\theta - 12\cos^2\theta + 1).
\end{aligned}$$

□

8. Write down the primitive 6^{th} roots of unity and the primitive 8^{th} roots of unity in standard form $a + bi$.

The primitive 6^{th} roots of unity are

$$e^{2\pi i/6} \text{ and } e^{10\pi i/6}$$

—that is,

$$\frac{1}{2} + \frac{\sqrt{3}}{2}i \text{ and } \frac{1}{2} - \frac{\sqrt{3}}{2}i.$$

The primitive 8^{th} roots of unity are

$$e^{2\pi i/8} \text{ and } e^{6\pi i/8} \text{ and } e^{10\pi i/8} \text{ and } e^{14\pi i/8}$$

—that is,

$$\frac{1}{\sqrt{2}}(1+i) \text{ and } \frac{1}{\sqrt{2}}(-1+i) \text{ and } \frac{1}{\sqrt{2}}(-1-i) \text{ and } \frac{1}{\sqrt{2}}(1-i)$$

—that is,

$$\pm\frac{1}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}i.$$

9. Let $\phi := \cos(2\pi/5) + i\sin(2\pi/5)$, a primitive 5^{th} root of 1. Define $\alpha := \phi + \phi^4$, $\beta := \phi^2 + \phi^3$.

(i) Show that α and β are real numbers.

(ii) Show that $\alpha + \beta = -1$ and $\alpha\beta = -1$ (so that α, β are the roots of $x^2 + x - 1 = 0$).

(iii) Deduce that $\cos(2\pi/5) = \frac{1}{4}(\sqrt{5} - 1)$.

(i) <u>Claim</u> α is real.

<u>Proof</u> Note that $\phi = e^{\frac{2\pi i}{5}}$, and $\phi^4 = e^{\frac{8\pi i}{5}} = e^{-\frac{2\pi i}{5}} = \overline{\phi}$,
so $\alpha = \phi + \phi^4 = \phi + \overline{\phi} = 2\,\mathrm{Re}(\phi)$ is real. □

<u>Claim</u> β is real.

<u>Proof</u> Note that $\phi^3 = \overline{\phi^2}$, so much the same argument works again. □

(ii) <u>Claim</u> $\alpha + \beta = -1$

<u>Proof</u> Since ϕ is a primitive 5^{th} root of 1, we see that $\phi^4 + \phi^3 + \phi^2 + \phi + 1 = 0$ (this comes from factorising $\phi^5 - 1 = 0$ and noting that $\phi \neq 1$ because it is primitive).

That is, $\alpha + \beta + 1 = 0$. □

<u>Claim</u> $\alpha\beta = -1$

<u>Proof</u> Remembering that $\phi^5 = 1$, we have

$$\begin{aligned}(\phi + \phi^4)(\phi^2 + \phi^3) &= \phi^3 + \phi^6 + \phi^4 + \phi^7 \\ &= \phi^3 + \phi + \phi^4 + \phi^2 \\ &= -1\end{aligned}$$

using the observation in the previous claim. □

(iii) <u>Claim</u> $\cos(2\pi/5) = \frac{1}{4}(\sqrt{5} - 1)$

<u>Proof</u> By (ii), we know that α and β are the two roots of the quadratic $(z - \alpha)(z - \beta) = z^2 - (\alpha + \beta)z + \alpha\beta = z^2 + z - 1$.

But the roots of this quadratic are $\frac{-1 \pm \sqrt{5}}{2}$. Note that one of these roots is positive and one is negative. We want to determine which of the two is α.

Now $\phi + \phi^4$ and $\phi^2 + \phi^3$ are both real, and we have $\phi + \phi^4 = 2\cos\left(\frac{2\pi}{5}\right)$ and $\phi^2 + \phi^3 = 2\cos\left(\frac{4\pi}{5}\right)$. Since $\frac{\pi}{2} < \frac{4\pi}{5} < \pi$, we see that $\cos\left(\frac{4\pi}{5}\right) < 0$, so

$$\cos\left(\frac{2\pi}{5}\right) = \frac{1}{2}(\phi + \phi^4) = \frac{-1 + \sqrt{5}}{4}.$$

□

10. Find the square roots of $-7+24i$. Now solve the equation $z^2-(2+2i)z+(7-22i)=0$.

(i) To find the square roots of $-7+24i$, let's write it in modulus-argument form.

We have $|-7+24i| = \sqrt{7^2+24^2} = 25$

and if $\arg(-7+24i)=\theta$ then $\tan\theta = -\frac{24}{7}$.

So $-7+24i = 25e^{i\theta}$ where $\tan\theta = -\frac{24}{7}$.

Let $w = Re^{i\phi}$ be a square root of $-7+24i$. Then $w^2 = R^2e^{2i\phi} = -7+24i = 25e^{i\theta}$.

So $R=5$, and

$$-\frac{24}{7} = \tan\theta = \tan(2\phi) = \frac{2\tan\phi}{1-\tan^2\phi}.$$

For convenience, let $t=\tan\phi$. Then we can rearrange to a get a quadratic in t: we have $12t^2-7t-12=0$, and so

$$t = \tan\phi = \frac{7\pm\sqrt{7^2+4\cdot 144}}{24} = \frac{7\pm 25}{24}.$$

So $\tan\phi = \frac{4}{3}$ or $\tan\phi = -\frac{3}{4}$.

Combining this with $R=5$, we see that the possibilities for w are

$$3+4i \text{ and } -3-4i \text{ and } 4-3i \text{ and } -4+3i.$$

This is a disconcertingly large number of solutions (secretly we're expecting two, right?). What we've shown is that if w is a square root of $-7+24i$, then w must be on that list of four values. We have not shown that all four of these values are indeed square roots of w. And in fact a quick check will show that $(4-3i)^2 = (-4+3i)^2 = 7-24i$, so these are no good, but $\pm(3+4i)$ are indeed the square roots of $-7+24i$.

(ii) Now we wish to solve the quadratic equation $z^2-(2+2i)z+(7-22i)=0$. Using the quadratic formula, we see that the solutions are

$$\begin{aligned} z &= \frac{(2+2i)\pm\sqrt{(2+2i)^2-4(7-22i)}}{2} \\ &= \frac{(2+2i)\pm\sqrt{8i-28+88i}}{2} \\ &= \frac{(2+2i)\pm\sqrt{-28+96i}}{2} \\ &= (1+i)\pm\sqrt{-7+24i} \\ &= (1+i)\pm(3+4i) \end{aligned}$$

so the two solutions are $4+5i$ and $-2-3i$.

11. By considering the seventh roots of -1, show that

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}.$$

What is the value of

$$\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}?$$

Claim

$$\cos\frac{\pi}{7}+\cos\frac{3\pi}{7}+\cos\frac{5\pi}{7}=\frac{1}{2}.$$

Proof Let $\zeta=\cos\left(\frac{\pi}{7}\right)+i\sin\left(\frac{\pi}{7}\right)$. Then $\zeta^7=-1$ but $\zeta\neq-1$.

By de Moivre's theorem, $\zeta^k=\cos\left(\frac{\pi k}{7}\right)+i\sin\left(\frac{\pi k}{7}\right)$.

Now $\zeta^7+1=0$,

so $(\zeta+1)(\zeta^6-\zeta^5+\zeta^4-\zeta^3+\zeta^2-\zeta+1)=0$,

but $\zeta\neq-1$ so $\zeta^6-\zeta^5+\zeta^4-\zeta^3+\zeta^2-\zeta+1=0$.

Looking just at the real part of this, we have

$$\cos\left(\frac{6\pi}{7}\right)-\cos\left(\frac{5\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)-\cos\left(\frac{3\pi}{7}\right)+\cos\left(\frac{2\pi}{7}\right)-\cos\left(\frac{\pi}{7}\right)+1=0.$$

But

$$\cos\left(\frac{6\pi}{7}\right)=-\cos\left(\frac{\pi}{7}\right)\ \text{ and }\ \cos\left(\frac{5\pi}{7}\right)=-\cos\left(\frac{2\pi}{7}\right)\ \text{ and }\ \cos\left(\frac{4\pi}{7}\right)=-\cos\left(\frac{3\pi}{7}\right),$$

so this becomes

$$1=2\left(\cos\left(\frac{\pi}{7}\right)+\cos\left(\frac{3\pi}{7}\right)+\cos\left(\frac{5\pi}{7}\right)\right),$$

so

$$\cos\left(\frac{\pi}{7}\right)+\cos\left(\frac{3\pi}{7}\right)+\cos\left(\frac{5\pi}{7}\right)=\frac{1}{2}.$$

$\square$

Now

$$\cos\left(\frac{2\pi}{7}\right)+\cos\left(\frac{4\pi}{7}\right)+\cos\left(\frac{6\pi}{7}\right)=-\left(\cos\left(\frac{5\pi}{7}\right)+\cos\left(\frac{3\pi}{7}\right)+\cos\left(\frac{\pi}{7}\right)\right)=-\frac{1}{2}.$$

12. Show that $\sigma^8 = -1$ if and only if σ is a primitive 16^{th} root of 1. Find the roots of the equation $x^8 = -1$, and use them to write $x^8 + 1$ as the product of four quadratic factors with real coefficients.

(i) Claim $\sigma^8 = -1$ if and only if σ is a primitive 16^{th} root of 1.

Proof ($\Rightarrow$) Suppose that $\sigma^8 = -1$. Then $\sigma^{16} = (\sigma^8)^2 = 1$, so certainly σ is a 16^{th} root of 1.

Let d be the smallest positive integer such that $\sigma^d = 1$ (so $d \leqslant 16$). Then $16 = qd + r$ for some integers q and r with $0 \leqslant r < d$. Now

$$\sigma^r = \sigma^{16-qd} = \sigma^{16}(\sigma^d)^{-q} = 1.$$

But r is less than d, and d is the smallest such positive integer, so we must have $r = 0$. That is, d divides 16.

If $d \neq 16$, then we must have that d divides 8, say $8 = kd$ where k is an integer. Now $-1 = \sigma^8 = (\sigma^d)^k = 1$ which is impossible! So $d = 16$. So σ is a *primitive* 16^{th} root of 1.

($\Leftarrow$) Suppose that σ is a primitive 16^{th} root of 1. Then $\sigma^{16} = 1$, so $\sigma^8 = \pm 1$.

But σ is a *primitive* 16^{th} root of 1, so $\sigma^8 \neq 1$.

So we must have $\sigma^8 = -1$.

□

There are more direct ways to prove this statement. For example, you could compute the roots of $\sigma^8 = -1$ and the primitive 16^{th} roots of 1, and show that the two lists of numbers are the same. I've given a different argument here because this same strategy crops up in other places too—there are useful ideas here. Watch out for similar arguments in the Groups and Group Actions course, for example.

(ii) Let's find the roots of $x^8 = -1$ using the previous part: we know that they're the primitive 16^{th} roots of 1.

But the primitive 16^{th} roots of 1 are $e^{\frac{2\pi k i}{16}}$ where $1 \leqslant k \leqslant 16$ and the highest common factor of k and 16 is 1. *(Can you prove this? Go on, do it!)*

So the primitive 16^{th} roots of 1 (the roots of $x^8 = -1$) are $e^{\frac{2\pi k i}{16}}$ for $k \in \{1, 3, 5, 7, 9, 11, 13, 15\}$, that is,

$$e^{\frac{2\pi i}{16}} \text{ and } e^{\frac{6\pi i}{16}} \text{ and } e^{\frac{10\pi i}{16}} \text{ and } e^{\frac{14\pi i}{16}} \text{ and } e^{\frac{18\pi i}{16}} \text{ and } e^{\frac{22\pi i}{16}} \text{ and } e^{\frac{26\pi i}{16}} \text{ and } e^{\frac{30\pi i}{16}}.$$

(iii) Claim

$$x^8+1 = (x^2-2\cos\left(\frac{\pi}{8}\right)x+1)(x^2-2\cos\left(\frac{3\pi}{8}\right)x+1)(x^2-2\cos\left(\frac{5\pi}{8}\right)x+1)(x^2-2\cos\left(\frac{7\pi}{8}\right)x+1)$$

—a product of four quadratic factors with real coefficients.

Proof We've just found the roots of this polynomial: we know that

$$x^8 + 1 = (x - e^{\frac{\pi i}{8}})(x - e^{-\frac{\pi i}{8}})(x - e^{\frac{3\pi i}{8}})(x - e^{-\frac{3\pi i}{8}})(x - e^{\frac{5\pi i}{8}})(x - e^{-\frac{5\pi i}{8}})(x - e^{\frac{7\pi i}{8}})(x - e^{-\frac{7\pi i}{8}})$$

and combining these pairs of linear factors gives the quadratics as claimed. □