INTRODUCTION TO UNIVERSITY MATHEMATICS – SHEET 2

(Exercises on lectures 5-8)

1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$ for all real numbers x.

(i) What are $f([0,\pi])$, $f([0,2\pi])$, $f([0,3\pi])$?

(ii)What are $f^{-1}(\{0\})$, $f^{-1}(\{1\})$, $f^{-1}(\{2\})$?

(iii) Let $A = [0,\pi]$ and $B = [2\pi, 3\pi]$. Show that $f(A \cap B) \neq f(A) \cap f(B)$.

(iv) Let $A = [0,\pi]$. Find $f(A)$, $f^{-1}(f(A))$, and $f(f^{-1}(A))$.

2. In each of the following cases, either find a function $f\colon \mathbb{R} \to \mathbb{R}$ with the required properties, or show that no such function exists.

(i) $f((0,1)) = \mathbb{R}$ and $f^{-1}(\mathbb{R}) = (0,1)$.

(ii) $f((0,1)) = (0,2)$ and $f((0,2)) = (0,1)$.

(iii) $f((0,1)) = (0,2)$ and $f^{-1}((0,2)) = (0,1)$.

(iv) $f((0,1)) = (0,2)$ and $f^{-1}((0,2)) = (0,2)$.

(v) $f((0,1)) = (0,1)$ and $f([0,1]) = (0,2)$.

3. Let X, Y be sets with $A, B \subseteq X$ and let $C,\ D \subseteq Y$, For $f\colon X \to Y$ prove that:

(i) $f(A) \cup f(B) = f(A \cup B)$.

(ii) $f^{-1}(C) \cup f^{-1}(D) = f^{-1}(C \cup D)$.

(iii) $f^{-1}(C) \cap f^{-1}(D) = f^{-1}(C \cap D)$.

4. Let R, S and T be sets and $f\colon R \to S$ and $g\colon S \to T$ be functions. Which of the following statements are true, which false? In each case either carefully prove the statement or give a specific counter-example.

(i) if f and g are injective then $g \circ f$ is injective.

(ii) if g is surjective then $g \circ f$ is surjective.

(iii) if $g \circ f$ is surjective then g is surjective.

5. Let $A = \{1,\ 2, 3\}$ and $B = \{1,\ 2,\ 3,\ 4, 5\}$.

(i) How many $f\colon A \to B$ are there?

(ii) How many $f\colon B \to A$ are there?

(iii) How many injective $f\colon A \to B$ are there?

(iv) How many injective $f\colon B \to A$ are there?

(v) How many surjective $f\colon A \to B$ are there?

(vi) How many surjective $f\colon B \to A$ are there?

6. Which of the following statements about natural numbers are true, which false?

(i) 2 is prime or 2 is odd.

(ii) 2 is prime or 2 is even.

(iii) If 2 is odd then 2 is prime.

(iv) If 2 is even then 2 is prime.

(v) For all $n \in \mathbb{N}$, if n is a square number then n is not prime.

(vi) For all $n \in \mathbb{N}$, n is not prime if and only if n is a square number.

(vii) For all even primes $p > 2$, $p^2 = 2019$.

7. Let P and Q be two statements. (For example, P might be '$x > 2$' and Q might be '$x^2 > 3$'. In this case $P \Rightarrow Q$. The negation $\neg P$ would be '$x \leqslant 2$'.) For two general statements P and Q which of the following sentences are equivalent? (The expression 'is true' has been suppressed in many of the sentences. You should obtain five different groups.)

(a)	P implies Q,	(h)	$P \Rightarrow \neg Q$,
(b)	Q is sufficient for P to be true,	(i)	P is necessary and sufficient for Q,
(c)	Q if P,	(j)	$\neg Q$ only if P,
(d)	P only if Q,	(k)	if Q is true then P is false,
(e)	$\neg P$ if and only if $\neg Q$,	(l)	P is false only if Q is false,
(f)	$P \Rightarrow Q$,	(m)	$\neg P \Leftarrow Q$.
(g)	$\neg P$ is necessary for $\neg Q$,	(n)	P is implied by Q being false

PTO

8. Critique each of the following "proofs". Firstly, is the statement correct? If not can you amend the statement into something that is true? Secondly, if the statement is correct, is the proof correct – are there genuine falsehoods or false deductions in the proof? Thirdly, if the proof is correct could it be improved on (in terms of clarity, order, concision)?

(i) 1 *is the largest natural number.* Let n denote the largest natural number. If $n > 1$ then $n^2 > n$ and as n^2 is also a natural number this contradicts the maximality of n. Hence $n \leqslant 1$ and so $n = 1$.

(ii) *The sum*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

is rational. We shall prove this by induction. Certainly 1 is rational and for $n \geqslant 1$ if

$$q_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n}$$

is rational then $q_{n+1} = q_n + (-1)^n/(n+1)$ is rational. The proof follows by induction.

(iii) *Every bounded differentiable function* $f\colon \mathbb{R} \to \mathbb{R}$ *is constant.* By assumption there exist real numbers M and N such that $M \leqslant f(x) \leqslant N$ for all x. Taking derivatives we get that $0 \leqslant f'(x) \leqslant 0$ for all x. If $f'(x) = 0$ for all x then f is constant.

(iv) *Let* $f\colon X \to Y$ *and* $g\colon Y \to Z$ *be functions. If* $g \circ f$ *is 1–1 then* g *is 1–1.* Let y_1 and y_2 be distinct points of Y. Choose distinct points $x_1, x_2 \in X$ such that $f(x_i) = y_i$. As $g \circ f$ is 1–1 then $g \circ f(x_1) = g(y_1)$ and $g \circ f(x_2) = g(y_2)$ are distinct. Hence g is 1–1.

A function $f\colon \mathbb{R} \to \mathbb{R}$ has properties C, L and U if

(C):	$\forall \varepsilon > 0 \quad \forall a \in \mathbb{R} \quad \exists \delta > 0 \quad$ whenever $\lvert x - a\rvert < \delta$ then $\lvert f(x) - f(a)\rvert < \varepsilon.$
(L):	$\exists K > 0 \quad \forall x, y \in \mathbb{R} \quad \lvert f(x) - f(y)\rvert \leqslant K \lvert x - y\rvert .$
(U):	$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall a \in \mathbb{R} \quad$ whenever $\lvert x - a\rvert < \delta$ then $\lvert f(x) - f(a)\rvert < \varepsilon.$

(v) *Property U implies property C.* This is obvious because the δ that works in U for given ε and all a then works in C for given ε and particular a.

(vi) *Property L implies property U.* Let $\varepsilon > 0$ and set $\delta = \varepsilon/K$. Let $a \in \mathbb{R}$ and let $x \in \mathbb{R}$ be such that $\lvert x - a\rvert < \delta$. Then by property L

$$\lvert f(x) - f(a)\rvert \leqslant K \lvert x - a\rvert = K(\varepsilon/K) = \varepsilon.$$

Hence property U follows.