INTRODUCTION TO UNIVERSITY MATHEMATICS – SHEET 1

(Exercises on lectures 1-4)

1. Given n positive numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n \leqslant 1/3$, prove by induction that

$$(1 - x_1)(1 - x_2) \times \cdots \times (1 - x_n) \geqslant 2/3.$$

2. Using the definition of addition given in lectures, show that addition is commutative on $\mathbb{N}$. That is, $x + y = y + x$ for all natural numbers x, y.

(i) Prove this first for $y = 0$ and all x by inducting on x.
(ii) Prove the result for $y = 1$ and all x.
(iii) Prove the general result by inducting on y.

3. Let x and b be integers with $x > 0$ and $b \geqslant 2$. Show that there are unique integers $a_0, a_1, a_2, \ldots$ with $0 \leqslant a_i < b$ such that

$$x = a_0 + a_1 b + a_2 b^2 + \cdots .$$

This is called the *expansion of x in base b*. [You may assume without proof that there is a non-negative integer n such that $b^n \leqslant x < b^{n+1}$.]

4. (i) Let A, B, C be subsets of a set X. Write out a proof that

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

(ii) Let D, E, F be subsets of X. Use (i) and De Morgan's laws to show that

$$D \cup (E \cap F) = (D \cup E) \cap (D \cup F).$$

5. Recursively we can define the *cardinality* $|X|$ of a finite set X as follows:

- The cardinality of the empty set is 0.
- The cardinality of a set X is $n + 1$ if there exists $x \in X$ such that $|X \backslash \{x\}| = n$.

Let A and B be finite sets.
(i) Assume for this part that A and B are disjoint. Use induction on $|B|$ to show that $A \cup B$ is finite and that

$$|A \cup B| = |A| + |B| .$$

(ii) Show that $A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$. Deduce that

$$|A| + |B| = |A \cup B| + |A \cap B| .$$

(iii) (Optional) Comment on whether it is clear that the given definition of cardinality of a finite set is well-defined. Can you resolve any concerns you have with the definition?

6. Which of the following relations on $\mathbb{N}$ are reflexive, which are symmetric, which are transitive?
(i) the relation $a|b$ (read as 'a divides b').
(ii) the relation $a \nmid b$ (does not divide).
(iii) a, b are related if a, b leave the same remainder after division by 2019.
(iv) a, b are related if $\mathrm{hcf}(a, b) > 2019$.
[Here $\mathrm{hcf}(a, b)$ denotes the highest common factor, or greatest common divisor, of a and b. It is the largest natural number that divides both a and b.]

7. How many partitions are there of a set of size 1? of size 2? of size 3? of size 4? of size 5?

8. Let $S = \{(m, n) : m, n \in \mathbb{Z},\ n \geqslant 1\}$. Show that $\sim$, as defined by

$$(m_1, n_1) \sim (m_2, n_2) \quad \Leftrightarrow \quad m_1 n_2 = m_2 n_1$$

is an equivalence relation on S. Show further that $\oplus$ and $\otimes$, as defined by

$$\begin{aligned}(m_1, n_1) \oplus (m_2, n_2) &= (m_1 n_2 + m_2 n_1, n_1 n_2) \\ (m_1, n_1) \otimes (m_2, n_2) &= (m_1 m_2, n_1 n_2)\end{aligned}$$

are well-defined binary operations on the set of equivalence classes $S/\sim$. What is $S/\sim$ a model for?