M5 Fourier Series and PDEs

Course synopsis

Overview

While developing the theory of heat conduction in the early 19th century, Jean-Baptiste Joseph Fourier kick-started a mathematical revolution by claiming that "every" real-valued function defined on a finite interval could be expanded as an infinite series of elementary trigonometric functions — cosines and sines. The need for rigorous mathematical analysis to assess this astonishing claim led to a surprisingly large proportion of the material covered in prelims, part A and beyond (*e.g.* the definition of a function, the ε-δ definition of limit, the theory of convergence of sequences and series of functions, Lebesgue integration and Cantor's set theory). The implications of Fourier's claim for practical applications were no less powerful or far-ranging: the decomposition led to deep and fundamental insights into numerous physical phenomena (e.g. mass and heat transport, vibrations of elastic media, acoustics and quantum mechanics) and continue to be exploited today in numerous fields (e.g. signal processing, approximation theory and control theory).

In this course we introduce fundamental results for the pointwise convergence of Fourier's infinite trigonometric series — Fourier series. We then follow in Fourier's footsteps by using them to construct solutions to fundamental problems involving the heat equation, the wave equation and Laplace's equation — the three most ubiquitous partial differential equations in mathematics, science and engineering.

Reading list

[1] D. W. Jordan and P. Smith, *Mathematical Techniques* (Oxford University Press, 4th Edition, 2003)

[2] E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, 10th Edition, 1999)

[3] G. F. Carrier and C. E. Pearson, *Partial Differential Equations — Theory and Technique* (Academic Press, 1988)

Synopsis (14 lectures)

Fourier series: Periodic, odd and even functions. Calculation of sine and cosine series. Simple applications concentrating on imparting familiarity with the calculation of Fourier coefficients and the use of Fourier series. The issue of convergence is discussed informally with examples. The link between convergence and smoothness is mentioned, together with its consequences for approximation purposes.

Partial differential equations: Introduction in descriptive mode on partial differential equations and how they arise. Derivation of (i) the wave equation of a string, (ii) the heat equation in one dimension (box argument only). Examples of solutions and their interpretation. D'Alembert's solution of the wave equation and applications. Characteristic diagrams (excluding reflection and transmission). Uniqueness of solutions of wave and heat equations.

PDEs with Boundary conditions. Solution by separation of variables. Use of Fourier series to solve the wave equation, Laplace's equation and the heat equation (all with two independent variables). Laplace's equation in Cartesian and in plane polar coordinates. Applications.

Authorship and acknowledgments

The author of these notes is **Jim Oliver**, taken largely from notes originally written Ruth Baker, Yves Capdeboscq, Alan Day and Janet Dyson, and typeset by **Benjamin Walker**. Please email comments and corrections to the course lecturer.

Fourier Series & PDEs: Lectures 1-2

Motivation

Example: existence of a convergent Fourier series

- Recall
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
for $z \in \mathbb{C}$.

- If we let $z = e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$, then
$$\begin{aligned} \mathrm{Im}\,(e^z) &= \mathrm{Im}\left(e^{\cos\theta} e^{i\sin\theta}\right) = e^{\cos\theta}\sin(\sin\theta)\,, \\ \mathrm{Im}\,(z^n) &= \mathrm{Im}\left(e^{in\theta}\right) = \sin n\theta\,. \end{aligned}$$

- Hence, $e^{\cos\theta}\sin(\sin\theta) = \underbrace{\sum_{n=1}^{\infty} \frac{\sin n\theta}{n!}}_{\text{Fourier (sine) series}}$ for $\theta \in \mathbb{R}$.

- Question: How would you generate an example of a convergent Fourier cosine series?

- Answer: By taking instead the real part in the argument above.

- Question: More generally, which functions $f : \mathbb{R} \to \mathbb{R}$ can be expressed as a Fourier series?

Example: heat conduction

- Suppose $T(x,t)$ is such that

  (1) $T_t = T_{xx}$ for $0 < x < \pi\,,\ t > 0$,

  (2) $T(0,t) = 0$, $T(\pi,t) = 0$ for $t > 0$,

  (3) $T(x,0) = e^{\cos x}\sin(\sin x)$ for $0 < x < \pi$.

- Observe $T(x,t) = \sum_{n=1}^{N} b_n \sin(nx) e^{-n^2 t}$ satisfies (1) and (2) for all $b_1, b_2, \ldots, b_n \in \mathbb{R}$, $N \in \mathbb{N} \setminus \{0\}$.

- Question: how should we pick N and the constants b_n?

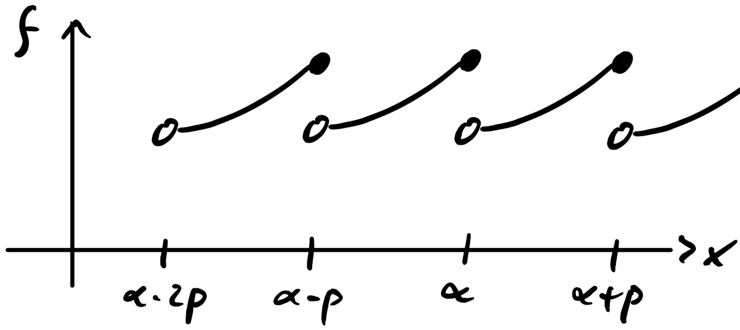
- Answer: $N = \infty$ and $b_n = \frac{1}{n!}$ to satisfy (3), i.e. a solution of the IBVP (1)–(3) is
$$T(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} \sin(nx) e^{-n^2 t}\,.$$

- Question: But what about other initial conditions?

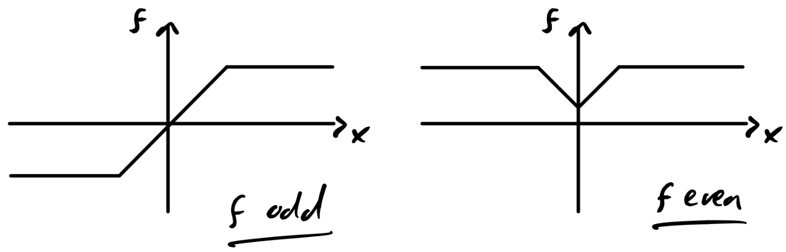
Periodic, even and odd functions

Definitions

- **Definition:** $f : \mathbb{R} \to \mathbb{R}$ is a periodic function if $\exists p > 0$ s.t. $f(x+p) = f(x) \forall x \in \mathbb{R}$. In this case p is a period for f and f is called **p–periodic**. A period is not unique, but if there exists a smallest such p it is called the prime period.
- Some examples:
  - $f =$ const. is p–periodic $\forall p > 0$, so has no prime period.
  - $\sin x$ has prime period 2π.
  - x and x^2 are not periodic.
- Note if f is periodic with prime period p then the graph of f repeats every p, e.g.



- $f : (\alpha, \alpha + p] \to \mathbb{R}$ can be extended uniquely to be p–periodic.
- **Definition:** The periodic extension $F : \mathbb{R} \to \mathbb{R}$ of $f : (\alpha, \alpha+p] \to \mathbb{R}$ is defined by $F(x) = f(x - mp)$, where for each $x \in \mathbb{R}$, m is the unique integer such that $x - mp \in (\alpha, \alpha + p]$.
- **Properties of periodic functions:** If f and g are p–periodic, then
  (1) f, g are np–periodic $\forall n \in \mathbb{N} \setminus \{0\}$;
  (2) $\alpha f + \beta g$ are p–periodic $\forall \alpha, \beta \in \mathbb{R}$;
  (3) fg is p–periodic;
  (4) $f(\lambda x)$ is p/λ–periodic $\forall \lambda > 0$;
  (5) $\int_0^p f(x)\,\mathrm{d}x = \int_\alpha^{\alpha+p} f(x)\,\mathrm{d}x \ \forall \alpha \in \mathbb{R}$.
- **Definition:** $f : \mathbb{R} \to \mathbb{R}$ is **odd** if $f(x) = -f(-x)\ \forall x \in \mathbb{R}$.
- **Definition:** $f : \mathbb{R} \to \mathbb{R}$ is **even** if $f(x) = f(-x)\ \forall x \in \mathbb{R}$.
- For example, x^n is odd for n odd and even for n even (hence the naming convention).
- Note symmetries of graphs of odd/even functions:



- **Properties of odd/even functions:** If f, f_1 are odd and g, g_1 are even, then:
  (1) $f(0) = 0$;
  (2) $\int\limits_{-\alpha}^{\alpha} f(x)\,\mathrm{d}x = 0 \ \forall \alpha \in \mathbb{R}$;
  (3) $\int\limits_{-\alpha}^{\alpha} g(x)\,\mathrm{d}x = 2\int\limits_{0}^{\alpha} g(x)\,\mathrm{d}x \ \forall x \in \mathbb{R}$;
  (4) fg is odd, ff_1 is even, and gg_1 is even.

Fourier series for functions of period 2π

- Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π. We want an expansion for f of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big). \qquad (\star)$$

- Question 1: If $(\star)$ is true, can we find the constants $a_n,\ b_n$ in terms of f?
- Question 2: With these a_n and b_n, when is $(\star)$ true?

Question 1

- Suppose $(\star)$ is true and that we can integrate it term by term, then

$$\int\limits_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 \int\limits_{-\pi}^{\pi} \mathrm{d}x + \sum_{n=1}^{\infty}\left(a_n \underbrace{\int\limits_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x}_{0} + b_n \underbrace{\int\limits_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x}_{0}\right),$$

  giving

$$a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\,\mathrm{d}x\,,$$

  i.e. $a_0/2$ is the mean of f over a period.

- **Lemma:** Let $m, n \in \mathbb{N} \setminus \{0\}$. Then we have the orthogonality relations:

$$\int\limits_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}, \quad \int\limits_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x = 0 \quad \int\limits_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn},$$

  where δ_{mn} is Kronecker's delta.

- The proof is on the first problem sheet.
- Fix $m \in \mathbb{N} \setminus \{0\}$, multiply $(\star)$ by $\cos(mx)$ and assume that the orders of summation and integration may be interchanged:

$$\begin{aligned}\int\limits_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x =& \frac{1}{2}a_0 \int\limits_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x \\ &+ \sum_{n=1}^{\infty}\left(a_n \int\limits_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x + b_n \int\limits_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x\right)\end{aligned}$$

giving

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \frac{1}{2}a_0 \cdot 0 + \sum_{n=1}^{\infty}(a_n\pi\delta_{mn} + b_n \cdot 0) = \pi a_m\,,$$

so that

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x \quad \text{for} \quad m \in \mathbb{N}\setminus\{0\}.$$

- Similarly, fix $m \in \mathbb{N}\setminus\{0\}$, multiply ($\star$) by $\sin(mx)$ and assume that the orders of summation and integration may be interchanged to obtain

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x \quad \text{for} \quad m \in \mathbb{N}\setminus\{0\}.$$

- **Definition:** Suppose f is 2π-periodic and such that the Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad (n \in \mathbb{N}), \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad (n \in \mathbb{N}\setminus\{0\})$$

exist. Then we write

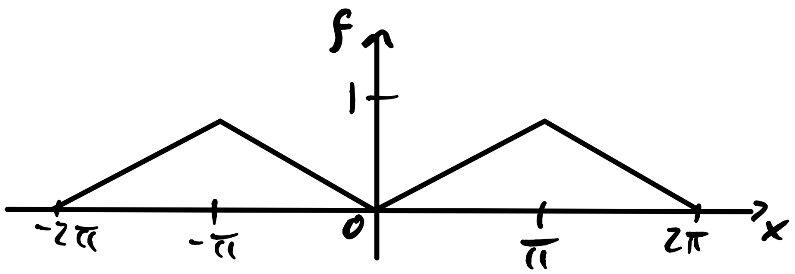
$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big),$$

where $\sim$ means the RHS is the Fourier series for f, regardless of whether or not it converges to f.

- Note the factor of $1/2$ in the first term of the Fourier series is for algebraic convenience.

Example 1

- Find the Fourier series for the 2π-periodic function f defined by $f(x) = |x|$ for $-\pi < x \leq \pi$.



- $f(x)$ even, so $f(x)\cos(nx)$ is even and $f(x)\sin(nx)$ is odd. Thus

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\,, \quad b_n = 0\,.$$

- Calculate

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x = \left[\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi} = \pi\,.$$

- For $n > 0$, we use integration by parts:

$$(uv)' = u'v + uv' \implies [uv]_a^b = \int_a^b u'v + uv'\,\mathrm{d}x.$$

- Pick $u = x$, $v = \frac{1}{n}\sin(nx)$, $a = 0$ and $b = \pi$ to give

$$\left[\frac{x}{n}\sin(nx)\right]_0^\pi = \int_0^\pi 1 \cdot \frac{1}{n}\sin(nx) + x\cos(nx)\,\mathrm{d}x\,.$$

- So

$$\int_0^\pi x\cos(nx)\,\mathrm{d}x = -\int_0^\pi \frac{1}{n}\sin(nx)\,\mathrm{d}x = \left[\frac{\cos(nx)}{n^2}\right]_0^\pi = \frac{(-1)^n - 1}{n^2}\,,$$

giving

$$a_n = -\frac{2}{\pi}\frac{[1-(-1)^n]}{n^2} = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\}\,, \\ -\dfrac{4}{\pi(2m+1)^2} & \text{for } n = 2m+1,\ m \in \mathbb{N}\,. \end{cases}$$

- Hence,

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2}\,.$$

□

Remarks

(1) Partial sums are defined by

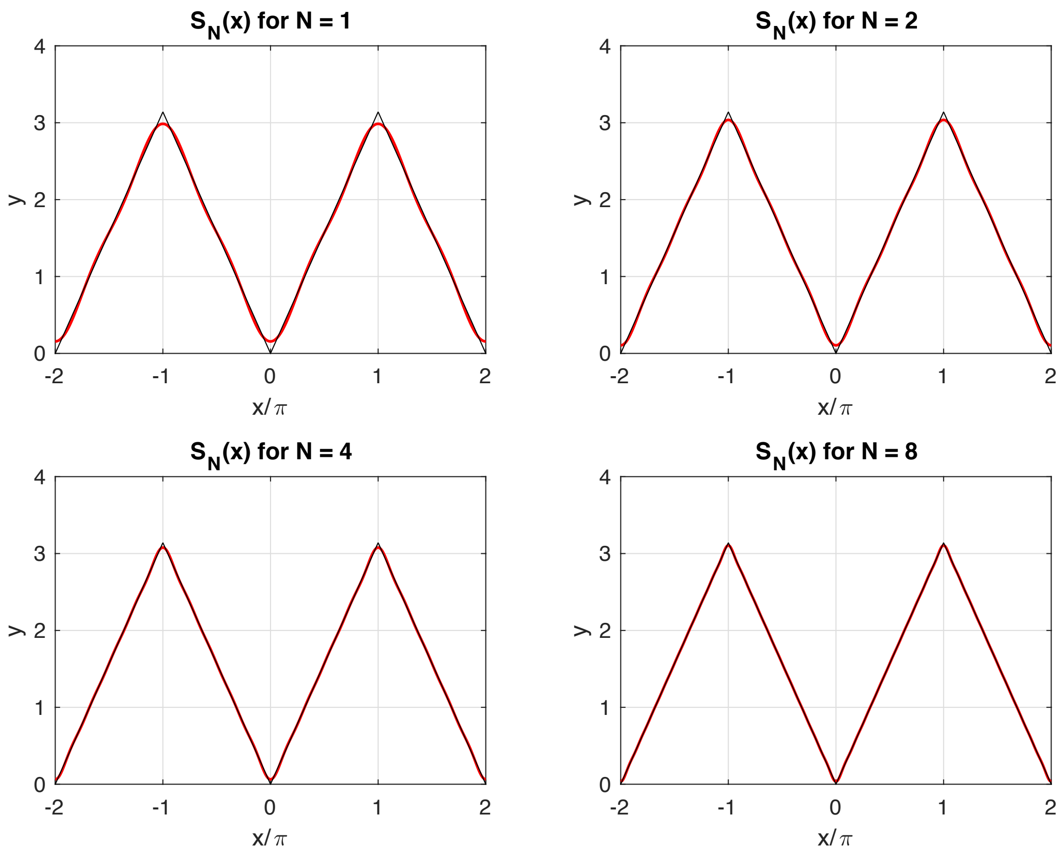
$$S_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{N}\frac{\cos((2m+1)x)}{(2m+1)^2}$$

for $N \in \mathbb{N}$. Plots below suggest that Fourier series converges on $\mathbb{R}$, i.e.

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \text{for } x \in \mathbb{R}\,. \qquad (\dagger)$$

(2) If this is true, we can pick x to evaluate the sum of a series, e.g. $x = 0$ gives

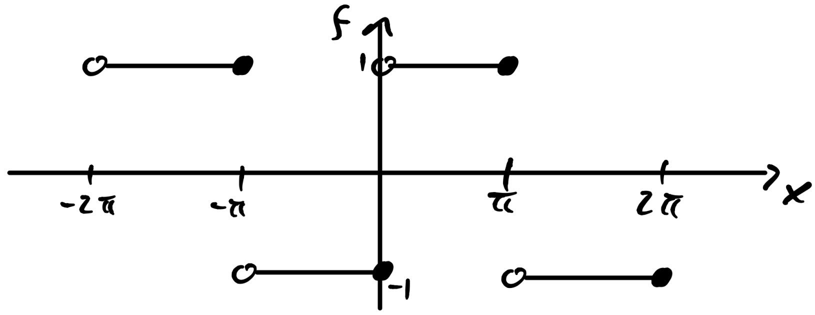
$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \implies \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}\,.$$



Example 2

- Find the Fourier Series for the 2π-periodic function f defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x \leq \pi\,, \\ -1 & \text{for } -\pi < x \leq 0\,. \end{cases}$$



- f is odd for $x \neq k\pi \;\; k \in \mathbb{Z}$, so

$$a_n = 0\,, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,\mathrm{d}x\,.$$

- $f(x) = 1$ for $0 < x < \pi$, so

$$b_n = \left[-\frac{2}{\pi}\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2[1-(-1)^n]}{\pi n}$$

- Hence

$$f(x) \sim \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin((2m+1)x)}{2m+1}\,.$$

□

Remarks

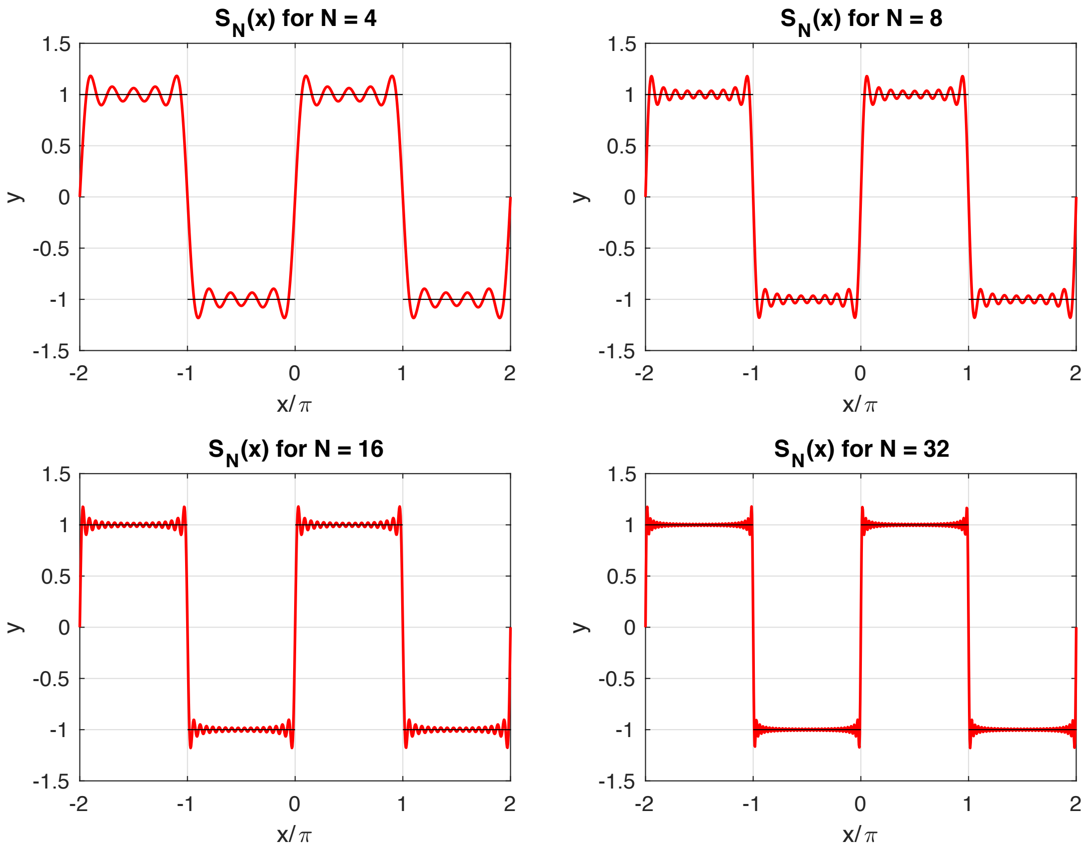
(1) Partial sums are defined by

$$S_N(x) = \frac{4}{\pi}\sum_{m=0}^{N}\frac{\sin((2m+1)x)}{2m+1} \qquad \text{for } N \in \mathbb{N}\,.$$

Plots below suggest that

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x) & \text{for } x \neq k\pi,\; k \in \mathbb{Z}\,, \\ 0 & \text{for } x = k\pi,\; k \in \mathbb{Z}\,. \end{cases} \tag{‡}$$

(2) Note slower convergence than in Example 1 and persistent overshoot near discontinuities of f — this is called Gibb's phenomenon (more to follow on this).



Sine and cosine series

- Let f be 2π-periodic and such that the Fourier coefficients exist.
- If $f(x)$ is odd, then $f(x)\cos(nx)$ is odd and $f(x)\sin(nx)$ is even, giving

$$a_n = 0, \ \ b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,\mathrm{d}x$$

  and thereby

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx),$$

  which is called a **Fourier sine series**.

- Note that this is also true if f is odd only for *e.g.* $x \neq k\pi, \ k \in \mathbb{Z}$.
- Similarly, if $f(x)$ is even, then we obtain the **Fourier cosine series**

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx),$$

  where

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,\mathrm{d}x\,.$$

Remark

- In the next lecture we will state a convergence theorem that may be be applied to show that both (†) and (‡) are indeed true.