Fourier Series & PDEs: Lectures 3-4

Convergence of Fourier series

- **Definition:** The right-hand limit of f at c is $f(c_+) = \lim\limits_{\substack{h\to 0\\ h>0}} f(c+h)$ if it exists.
- **Definition:** The left-hand limit of f at c is $f(c_-) = \lim\limits_{\substack{h\to 0\\ h<0}} f(c+h)$ if it exists.
- **Remarks:**
  (1) $f(c)$ need not be defined for $f(c_+)$ or $f(c_-)$ to exist.
  (2) The existence part is important, *e.g.* if $f(x) = \sin(1/x)$ for $x \neq 0$, then $f(0_\pm)$ do not exist.
  (3) f is continuous at c if and only if $f(c_-) = f(c) = f(c_+)$.
  (4) In Example 2, f is continuous for $x/\pi \in \mathbb{R}\backslash\mathbb{Z}$ with *e.g.* $f(0_\pm) = \pm 1$ and $f(\pi_\pm) = \mp 1$.
- **Definition:** f is piecewise continuous on $(a,b) \subseteq \mathbb{R}$ if there exists a finite number of points $x_1, \ldots, x_m \in \mathbb{R}$ with $a = x_1 < x_2 < \ldots < x_m = b$ such that
  (i) f is defined and continuous on (x_k, x_{k+1}) for all $k = 1, \ldots, m-1$;
  (ii) $f(x_{k+})$ exists for $k = 1, \ldots, m-1$;
  (iii) $f(x_{k-})$ exists for $k = 2, \ldots, m$.
- **Remarks:**
  (1) Note that f need not be defined at its exceptional points $x_1, \ldots, x_m$!
  (2) The functions in Examples 1 and 2 are piecewise continuous on any interval $(a,b) \subseteq \mathbb{R}$.
- **Fourier Convergence Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, with f and f' piecewise continuous on $(-\pi, \pi)$. Then, the Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad (n \in \mathbb{N}),$$

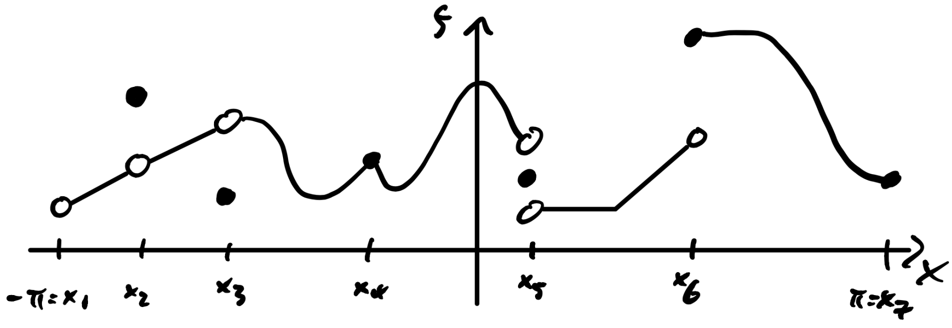
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad (n \in \mathbb{N}\setminus\{0\})$$

exist, and

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big) \quad \text{for} \quad x \in \mathbb{R}.$$

- **Remarks on the hypotheses:**
  (1) If f and f' are piecewise continuous on $(-\pi, \pi)$, then there exist $x_1, \ldots, x_m \in \mathbb{R}$ with $-\pi = x_1 < x_2 < \ldots < x_m = \pi$ such that
    (i) f and f' are continuous on (x_k, x_{k+1}) for $k = 1, \ldots, m-1$.
    (ii) $f(x_{k+})$ and $f'(x_{k+})$ exist for $k = 1, \ldots, m-1$.
    (iii) $f(x_{k-})$ and $f'(x_{k-})$ exist for $k = 2, \ldots, m$.

(2) Thus, in any period f, f' are continuous except possibly at a finite number of points. At each such point f' need not be defined, and one or both of f and f' may have a jump discontinuity, as illustrated for the various different possibilities in the schematic below



(3) For example, if

$$f(x) = \begin{cases} x^{1/2} & \text{for } 0 \leq x \leq \pi\,, \\ 0 & \text{for } -\pi < x < 0, \end{cases}$$

then

$$f'(x) = \begin{cases} \frac{1}{2}x^{-1/2} & \text{for } 0 < x < \pi\,, \\ 0 & \text{for } -\pi < x < 0\,, \\ \text{undefined} & \text{for } x = 0, \pi. \end{cases}$$

Hence, while f is piecewise continuous on $(-\pi, \pi)$, f' is not because $f'(0_+)$ does not exist.

- **Remarks on the convergence result:**

(1) The partial sums of the Fourier series are defined by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for} \quad N \in \mathbb{N}\backslash\{0\}.$$

The theorem states that the partial sums converge pointwise in the sense that

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\big(f(x_+) + f(x_-)\big) \quad \text{for each} \quad x \in \mathbb{R}.$$

(2) If f has a jump discontinuity at x so that $f(x_+) \neq f(x_-)$, then the Fourier series converges to $\big(f(x_+) + f(x_-)\big)/2$, *i.e.* the average of the left- and right-hand limits of f at x.

(3) If f is continuous at x so that $f(x_-) = f(x) = f(x_+)$, then the Fourier series converges to $f(x)$.

(4) If we redefined f to be equal to the average of its left- and right-hand limits at each of its jump discontinuities, then the Fourier series would converge instead to f on $\mathbb{R}$.

(5) If f is defined only on *e.g.* $(-\pi, \pi]$, then the Fourier Convergence Theorem holds for its 2π-periodic extension.

(6) We note that the Fourier Convergence Theorem implies that

$$\frac{1}{2}\big(g(x_+) + g(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx), \quad \text{for} \quad x \in \mathbb{R},$$

$$\frac{1}{2}\big(h(x_+) + h(x_-)\big) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{for} \quad x \in \mathbb{R},$$

where $g : \mathbb{R} \to \mathbb{R}$ is the <u>even part of f</u> and $h : \mathbb{R} \to \mathbb{R}$ is the <u>odd part of f</u>, defined by

$$g(x) = \frac{1}{2}\big(f(x) + f(-x)\big), \quad h(x) = \frac{1}{2}\big(f(x) - f(-x)\big) \quad \text{for} \quad x \in \mathbb{R}.$$

- **Remarks on the proof:** While the proof is not examinable, it is amenable to methods from Prelims Analysis as follows.

  (1) Use the integral expressions for the Fourier coefficients and properties of periodic, even and odd functions to manipulate the partial sums into the form

$$S_N(x) - \frac{1}{2}\big(f(x_+) + f(x_-)\big) = \int_0^\pi F(x,t) \sin\left[\left(N + \frac{1}{2}\right)t\right] \mathrm{d}t,$$

  where
$$F(x,t) = \frac{1}{\pi}\left(\frac{f(x+t) - f(x_+)}{t} + \frac{f(x-t) - f(x_-)}{t}\right)\left(\frac{t}{2\sin(t/2)}\right).$$

  (2) Use the Mean Value Theorem (of Analysis II) to show that $F(x,t)$ is a piecewise continuous function of t on $(0,\pi)$, and hence deduce from the Riemann-Lebesgue Lemma (of Analysis III) that

$$\int_0^\pi F(x,t) \sin\left[\left(N + \frac{1}{2}\right)t\right] \mathrm{d}t \to 0 \quad \text{as } N \to \infty.$$

- **Remarks on differentiability and integrability:**

  (1) The Fourier series can be integrated termwise under weaker conditions, e.g. if f is only 2π-periodic and piecewise continuous on $(-\pi,\pi)$, then the Fourier Convergence Theorem implies

$$\int_0^x f(s)\,\mathrm{d}s = \frac{1}{2}a_0 x + \sum_{n=1}^{\infty}\left(a_n \int_0^x \cos(ns)\,\mathrm{d}s + b_n \int_0^x \sin(ns)\,\mathrm{d}s\right) \quad \text{for} \quad x \in \mathbb{R}.$$

  Note that the integral on the LHS is 2π-periodic if and only if $a_0 = 0$.

  (2) However, we need stronger conditions to differentiate termwise, *e.g.* if f is 2π-periodic and continuous on $\mathbb{R}$ with both f' and f'' piecewise continuous on $(-\pi,\pi)$, then the Fourier Convergence Theorem implies

$$\frac{1}{2}\left(f'(x_+) + f'(x_-)\right) = \sum_{n=1}^{\infty}\left(a_n \frac{\mathrm{d}}{\mathrm{d}x}(\cos(nx)) + b_n \frac{\mathrm{d}}{\mathrm{d}x}(\sin(nx))\right) \quad \text{for} \quad x \in \mathbb{R}.$$

Examples 1 and 2 revisited

- Recall the 2π-periodic function of Example 1 which we defined by setting

$$f(x) = |x| \quad \text{for } -\pi < x \le \pi.$$

- We calculate
$$f'(x) = \begin{cases} 1 & \text{for } 0 < x < \pi, \\ -1 & \text{for } -\pi < x < 0, \\ \text{undefined} & \text{for } x = 0,\ \pi. \end{cases}$$

- Since both f and f' are piecewise continuous on $(-\pi,\pi)$, with f continuous on $\mathbb{R}$, the Fourier Convergence Theorem gives

$$\frac{\pi}{2}-\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos\left((2m+1)x\right)}{(2m+1)^2}=f(x)\quad\text{for}\quad x\in\mathbb{R}. \tag{1.1}$$

Note that LHS = RHS $\neq |x|$ for $|x|>\pi$.

- Since f is piecewise continuous on $(-\pi,\pi)$, we can integrate termwise to obtain

$$\frac{\pi x}{2}+\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin\left((2m+1)x\right)}{(2m+1)^3}=\int_0^x f(s)\,\mathrm{d}s\quad\text{for}\quad x\in\mathbb{R}. \tag{1.2}$$

Note that while LHS = RHS is not periodic, the function $\int_0^x f(s)-\frac{\pi}{2}\,\mathrm{d}s$ is 2π-periodic.

- We calculate

$$f''(x)=\begin{cases}0 & \text{for } 0<x<\pi,\\ 0 & \text{for } -\pi<x<0,\\ \text{undefined} & \text{for } x=0,\ \pi.\end{cases}$$

- Since f is continuous on $\mathbb{R}$ and both f' and f'' are piecewise continuous on $(-\pi,\pi)$, we can differentiate termwise to obtain

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin\left((2m+1)x\right)}{2m+1}=\frac{1}{2}\big(f(x_-)+f(x_+)\big)=\begin{cases}1 & \text{for } 0<x<\pi,\\ -1 & \text{for } -\pi<x<0,\\ 0 & \text{for } x=0,\ \pi.\end{cases} \tag{1.3}$$

- Note that the function to which this Fourier series converges is equal to the function considered in Example 2 for $x/\pi\in\mathbb{R}\backslash\mathbb{Z}$, which deals thereby with the convergence and termwise integration of the Fourier series of the function in Example 2; it remains to note that, since that function is not continuous on $\mathbb{R}$, its Fourier series cannot be differentiated termwise — try it!

Rate of convergence

- The smoother f, *i.e.* the more continuous derivatives it has, the faster the convergence of the Fourier series for f.
- If the first jump discontinuity is in the p^{th} derivative of f, with the convention that $p=0$ if there is a jump discontinuity in f, then typically the slowest decaying a_n and b_n decay like $1/n^{p+1}$ as $n\to\infty$.
- For example, $p=1$ in (1.1), $p=2$ in (1.2) and $p=0$ in (1.3).
- This is an extremely useful result in practice (*e.g.* for approximately 1% accuracy we need 100 terms for $p=0$, but only 10 terms for $p=1$) and for checking calculations (*e.g.* an erroneous contribution to a Fourier coefficient can be rapidly identified if it does decay fast enough for large n).
- We make the following two remarks with the caveat that they are beyond the scope of this course:

  (1) If the Fourier coefficients decay like $1/n^{p+1}$ as $n\to\infty$ with $p\geq 1$, then the Weierstrass M-test (of Analysis II) may be used to show that the Fourier series for f converges uniformly to f on any interval $(a,b)\subset\mathbb{R}$.

  (2) If the Fourier coefficients decay like $1/n$ as $n\to\infty$ (so that $p=0$), then the partial sums of the Fourier series for f do not converge uniformly on any interval containing a jump discontinuity. Remarkably, the form of the non-uniformity is universal for such functions, being characterized by Gibb's phenomenon, as we shall now describe.

Gibb's phenomenon

- This is the persistent overshoot in Example 2 near a jump discontinuity. It happens whenever a jump discontinuity exists.
- As the number of terms in the partial sum tends to ∞, the width of the overshoot region tends to 0 (by the Fourier Convergence Theorem), while the total height of the overshoot region approaches $\gamma|f(x_+) - f(x_-)|$, where

$$\gamma = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin x}{x}\,\mathrm{d}x \approx 1.18,$$

i.e. approximately a 9% overshoot top and bottom. This is awful for approximation purposes!

Functions of any period

- Suppose now $f : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $2L$, where L is a positive number, not necessarily equal to π.
- We want to develop the analogous results for the Fourier series for $f(x)$. Since this will involve a series in the trigonometric functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$, where n is a positive integer, we make the transformation

$$x = \frac{LX}{\pi}, \quad f(x) = g(X)$$

which defines a new function $g : \mathbb{R} \to \mathbb{R}$.

- For $X \in \mathbb{R}$, it follows that

$$\begin{aligned} g(X + 2\pi) &= f\left(\frac{L}{\pi}(X + 2\pi)\right) \\ &= f\left(\frac{LX}{\pi} + 2L\right) \\ &= f\left(\frac{LX}{\pi}\right) \\ &= g(X), \end{aligned}$$

where we used the fact that $g(X) = f(LX/\pi)$ in the first equality; the fact that f is $2L$-periodic in the third equality; and the fact that $f(x) = g(LX/\pi)$ in the third equality. Thus, g is 2π-periodic, and we can use the transformation to derive the Fourier theory for f from that for g above.

- In particular, if we can write

$$g(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nX) + b_n \sin(nX)\big),$$

so that the Fourier coefficients a_n and b_n exist, then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\cos(nX)\,\mathrm{d}X, \\ &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(X)\sin(nX)\,\mathrm{d}X, \\ &= \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x, \\ &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x. \end{aligned}$$

- So if we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right),$$

then

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

- We wrap these formal calculations into the definition of the Fourier series for f.
- **Definition:** Suppose f is $2L$-periodic and such that the Fourier coefficients

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}), \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}\setminus\{0\}) \end{aligned}$$

exist. Then we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right),$$

where $\sim$ means the RHS is the Fourier series for f, regardless of whether or not it converges to f.

- **Remark:** The formulae for the Fourier coefficients may also be derived from the Fourier series for f by assuming that the orders of summation and integration may be interchanged and using the orthogonality relations

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = L\delta_{mn}$$

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = 0,$$

$$\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = L\delta_{mn}$$

where $n, m \in \mathbb{N}\setminus\{0\}$.

- We are now in a position to write down the corresponding Fourier Convergence Theorem.

- **Fourier Convergence Theorem:** Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic, with f and f' piecewise continuous on $(-L, L)$. Then, the Fourier coefficients

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}), \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x \quad (n \in \mathbb{N}\setminus\{0\}) \end{aligned}$$

exist, and

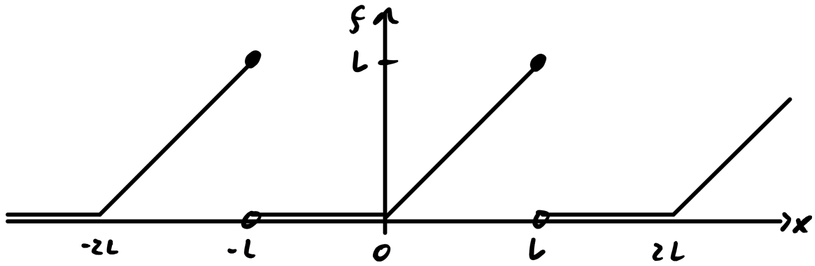
$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) \quad \text{for} \quad x \in \mathbb{R}.$$

Example 3

- Consider the $2L$-periodic function f defined by

$$f(x) = \begin{cases} x & \text{for } 0 < x \leq L\,, \\ 0 & \text{for } -L < x \leq 0\,. \end{cases}$$

Find the Fourier series for f and the function to which the Fourier series converges.



- By the definition of f, the Fourier coefficients are given by

$$a_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x\,, \quad b_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\,.$$

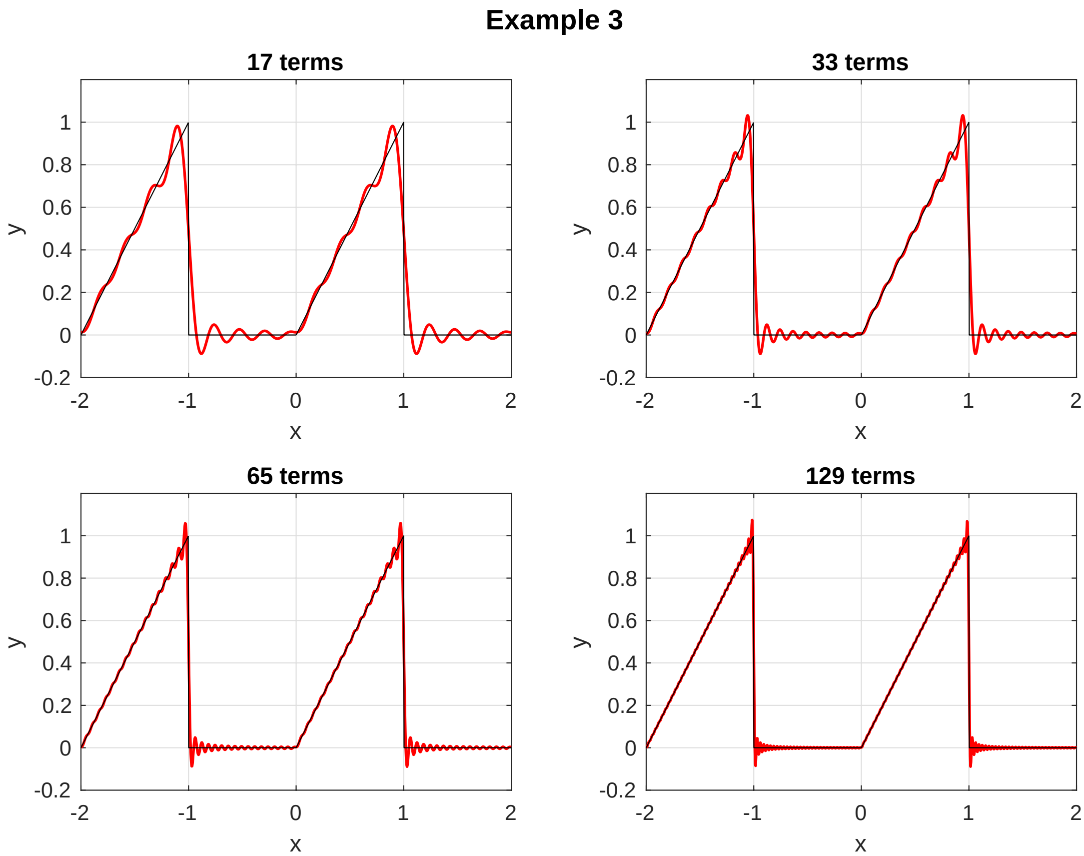
- A direct integration gives $a_0 = L/2$, but for $n \in \mathbb{N}\setminus\{0\}$ it is a bit quicker to evaluate

$$\begin{aligned} a_n + ib_n &= \frac{1}{L}\int_0^L \underbrace{x}_{u}\ \underbrace{\exp\left(\frac{in\pi x}{L}\right)}_{v'}\mathrm{d}x \\ &= \left[\frac{1}{L}\ \underbrace{x}_{u}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\right]_0^L - \frac{1}{L}\int_0^L \underbrace{1}_{u'}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\mathrm{d}x \\ &= -\left[\frac{1}{L}\left(\frac{L}{in\pi}\right)^2\exp\left(\frac{in\pi x}{L}\right)\right]_0^L + \frac{L}{in\pi}\exp\left(in\pi\right) \\ &= \frac{L}{n^2\pi^2}\left((-1)^n - 1\right) + \frac{iL(-1)^{n+1}}{n\pi}. \end{aligned}$$

- Thus
$$f(x) \sim \frac{L}{4} + \sum_{m=1}^{\infty} \left(-\frac{2L}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{L}\right) + \frac{L(-1)^{m+1}}{m\pi} \sin\left(\frac{m\pi x}{L}\right) \right).$$

- Since f and f' are piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f converges to $f(x)$ at points of continuity of f, *i.e.* for $x \neq (2k+1)L$, $k \in \mathbb{Z}$, while at the jump discontinuities the Fourier series converges to the average of the left- and right-hand limits of f, *i.e.* to $\big(f(L_+) + f(L_-)\big)/2 = (0 + L)/2 = L/2$ for $x = (2k+1)L$, $k \in \mathbb{Z}$.

- We note that the slowest decaying Fourier coefficients b_n decay as expected like $1/n$ as $n \to \infty$ because f has jump discontinuities so that $p = 0$. The plots below for $L = 1$ illustrate the slow convergence of the partial sums of the Fourier series, which is hindered by Gibb's phenomenon at the jump discontinuities.



Cosine and sine series

- In many practical applications we wish to express a given function $f : [0, L] \to \mathbb{R}$ in terms of either a Fourier cosine series or a Fourier sine series.

- This may be accomplished by extending f to be even (for only cosine terms) or odd (for only sine terms) on $(-L, 0) \cup (0, L)$ and then extending to a periodic function of period $2L$.

- We wrap these extensions and the corresponding Fourier series into the following definitions.

- **Definition:** The even $2L$-periodic extension $f_e : \mathbb{R} \to \mathbb{R}$ of $f : [0, L] \to \mathbb{R}$ is defined by
$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \leq x \leq L\,, \\ f(-x) & \text{for } -L < x < 0\,, \end{cases}$$

with $f_e(x+2L) = f_e(x)$ for $x \in \mathbb{R}$. The Fourier cosine series for $f : [0,L] \to \mathbb{R}$ is the Fourier series for f_e, *i.e.*

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad (n \in \mathbb{N}).$$

- **Definition:** The odd $2L$-periodic extension $f_o : \mathbb{R} \to \mathbb{R}$ of $f : [0,L] \to \mathbb{R}$ is defined by

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L\,, \\ -f(-x) & \text{for } -L < x < 0\,, \end{cases}$$

with $f_o(x+2L) = f_o(x)$ for $x \in \mathbb{R}$. The Fourier sine series for $f : [0,L] \to \mathbb{R}$ is the Fourier series for f_o, *i.e.*

$$f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \quad (n \in \mathbb{N}\backslash\{0\}).$$

- **Remarks:**

(1) Note that $f_o(x)$ is odd for $x/L \in \mathbb{R}\backslash\mathbb{Z}$ and odd (on $\mathbb{R}$) if and only if $f(0) = f(L) = 0$.

(2) Note that if f is continuous on $[0,L]$ and f' piecewise continuous on $(0,L)$, then the Fourier Convergence Theorem implies that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f_e(x) \text{ for } x \in \mathbb{R}\,,$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} f_o(x) & \text{for } x/L \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{for } x/L \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

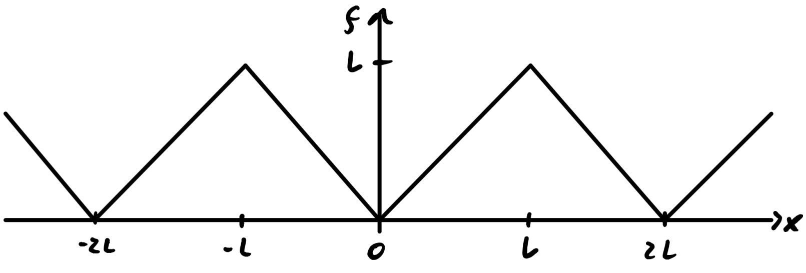
Example 4

- Consider the function $f : [0,L] \to \mathbb{R}$ defined by $f(x) = x$ for $0 \le x \le L$. Find the Fourier cosine and sine series for f and the functions to which each of them converge on $[0,L]$. Which truncated series gives the best approximation to f on $[0,L]$?

- The even $2L$-periodic extension f_e is defined by

$$f_e(x) = \begin{cases} x & \text{for } 0 \le x \le L\,, \\ -x & \text{for } -L < x < 0\,, \end{cases}$$

i.e. $f_e(x) = |x|$ for $-L < x \le L$.



- Since

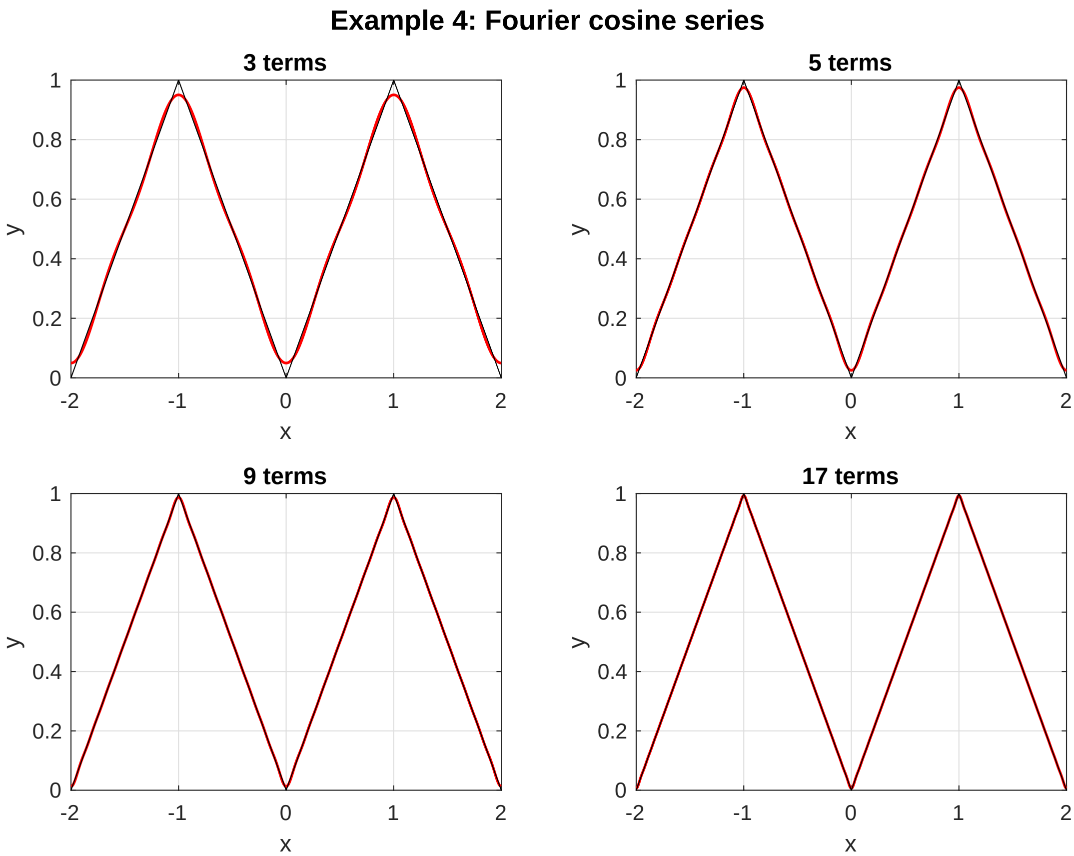
$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x\,,$$

an integration by parts yields the Fourier cosine series

$$f_e(x) \sim \frac{L}{2} - \sum_{m=0}^{\infty} \frac{4L}{(2m+1)^2\pi^2}\cos\left(\frac{(2m+1)\pi x}{L}\right).$$

- Since f_e is continuous on $\mathbb{R}$ and f_e' is piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f_e converges to f_e on $\mathbb{R}$.

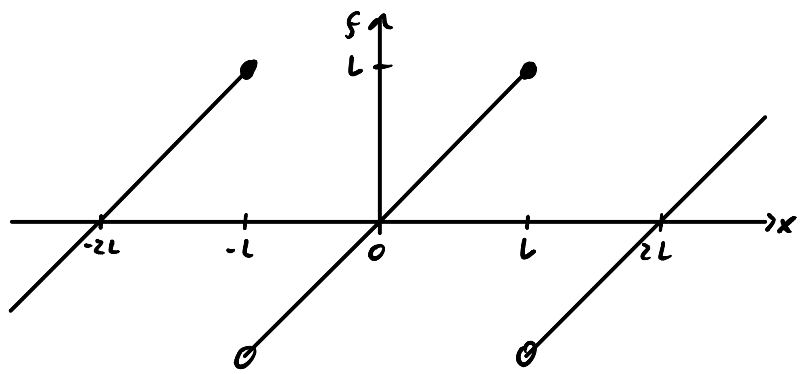
- Hence the Fourier cosine series for f converges to f on $[0, L]$, as illustrated by the plots below of the partial sums for $L = 1$.



- Similarly, the odd $2L$-periodic extension f_e is defined by

$$f_o(x) = \begin{cases} x & \text{for } 0 \le x \le L\,, \\ -(-x) & \text{for } -L < x < 0\,, \end{cases}$$

i.e. $f_o(x) = x$ for $-L < x \le L$.



- Since

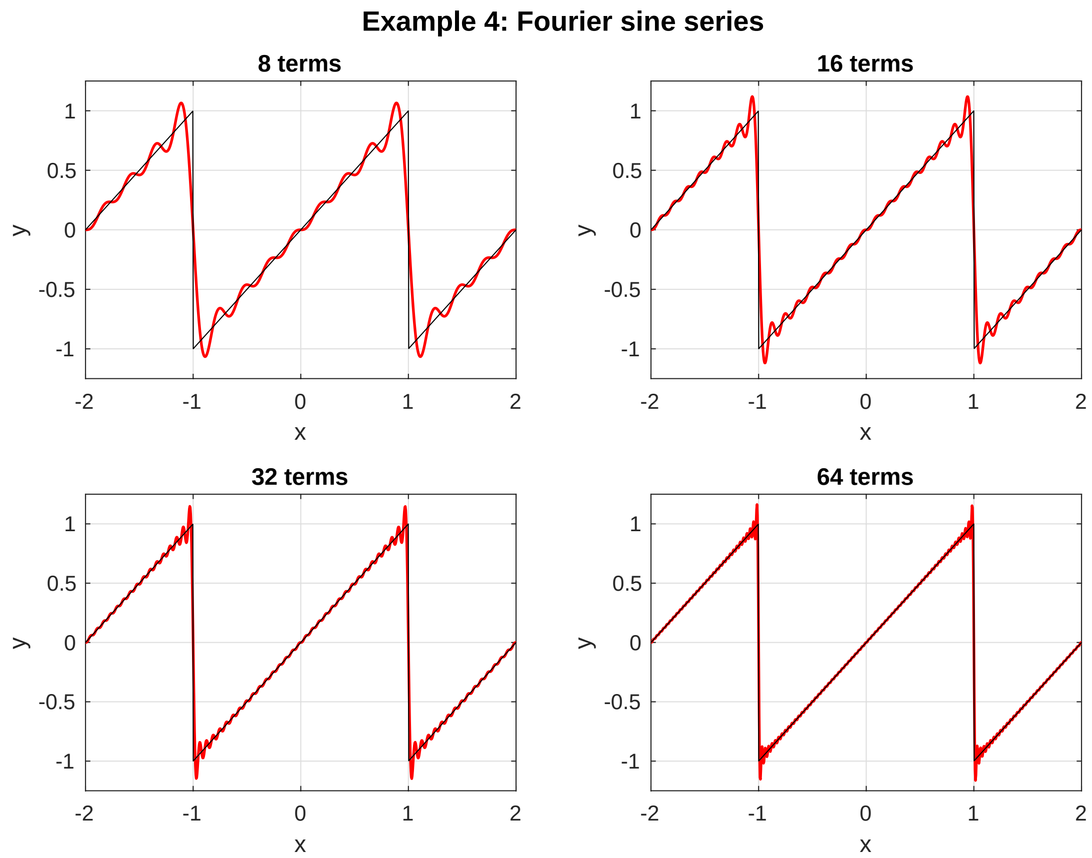
$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x\,,$$

an integration by parts yields the Fourier sine series

$$f_o(x) \sim \sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right).$$

- Since f_o and f_o' are piecewise continuous on $(-L, L)$, the Fourier Convergence Theorem implies that the Fourier series for f_o converges to $f_o(x)$ at points of continuity of f_o, *i.e.* for $x \neq (2k+1)L$, $k \in \mathbb{Z}$, while at the jump discontinuities the Fourier converges to the average of the left- and right-hand limits of f_0, *i.e.* to $\big(f(L_+) + f(L_-)\big)/2 = (-L + L)/2 = 0$ for $x = L$ and hence for $x = (2k+1)L$, $k \in \mathbb{Z}$.

- Hence, the Fourier sine series for f converges to $f(x)$ for $0 \leq x < L$, but to 0 for $x = L$, with Gibb's phenomenon again slowing the rate of convergence near the jump discontinuities as illustrated by the plots below of the partial sums for $L = 1$.



- The truncated cosine series gives a better approximation to f on $[0, L]$ than the truncated sine series because

  (1) it converges everywhere on $[0, L]$;

  (2) it converges more rapidly;

  (3) it does not exhibit Gibb's phenomenon.

- Finally, we note that f_e is equal to twice the even part of the function in Example 3, while f_o is equal to twice the odd part of the function in Example 3, which explains the rate of decay of the Fourier coefficients in Example 3.