Fourier Series & PDEs: Lectures 5-6

The PDEs we shall study

- We shall study the following partial differential equations.

PDE	Name	Unknown	Parameters
$T_t = \kappa T_{xx}$	Heat equation	$T(x,t)$	$\kappa > 0$
$y_{tt} = c^2 y_{xx}$	Wave equation	$y(x,t)$	$c > 0$
$T_{xx} + T_{yy} = 0$	Laplace's equation	$T(x,y)$	None

- We shall derive each of them using physical principles and develop methods to solve several physically important problems formed by imposing appropriate boundary conditions and/or initial conditions — different for each of them!

Some preliminaries

- **Fundamental Theorem of Calculus:** If $f(x)$ is continuous in a neighbourhood of a, then

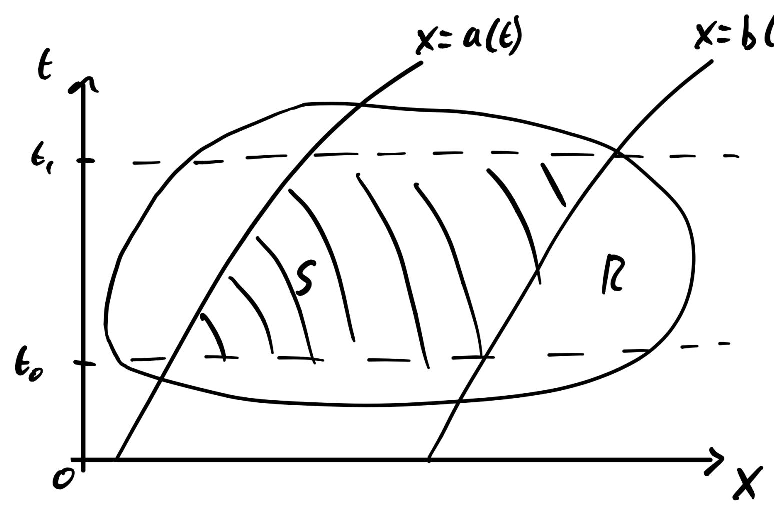
$$\frac{1}{h}\int_{a}^{a+h} f(x)\,\mathrm{d}x \to f(a) \quad \text{as} \quad h \to 0.$$

- **Leibniz's Integral Rule:** Let $F(x,t)$ and $\partial F/\partial t$ be continuous in both x and t in some region R of the (x,t) plane containing the region $S = \{(x,t) : a(t) \leq x \leq b(t),\ t_0 \leq t \leq t_1\}$, where the functions $a(t)$ and $b(t)$ and their derivatives are continuous for $t \in [t_0, t_1]$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a(t)}^{b(t)} F(x,t)\ \mathrm{d}x = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}(x,t)\,\mathrm{d}x + \dot{b}(t)F(b(t),t) - \dot{a}(t)F(a(t),t).$$

As a result, if $a(t)$ and $b(t)$ are constants, then

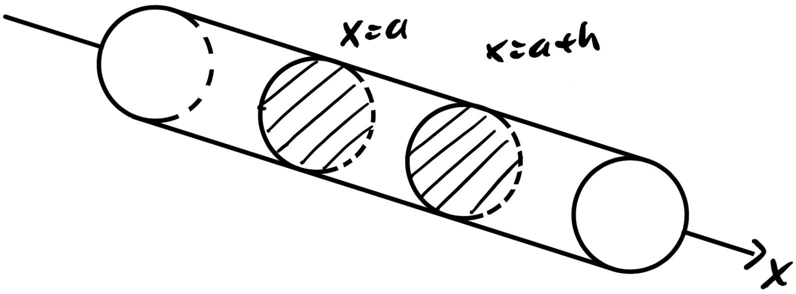
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{a}^{b} F(x,t)\ \mathrm{d}x = \int_{a}^{b} \frac{\partial F}{\partial t}(x,t)\,\mathrm{d}x.$$



The heat equation

Derivation in one dimension

- Consider a rigid isotropic conducting rod (*e.g.* metal) of constant cross-sectional area A lying along the x-axis. We shall consider conservation of thermal or heat energy in the arbitrary section of the rod in $a \leq x \leq a + h$, where a and h are constants. The geometric setup is as illustrated in the schematic below.



- In the simplest one-dimensional model we assume that the lateral surfaces of the rod are insulated, so that no thermal energy can be transported through them and the absolute temperature T may be taken to be a function of distance x along an axis of the rod and time t. This model is applicable if the rod is long and thin, like a wire.

- We denote by ρ the density of the rod and by c the specific heat of the rod, and we assume that these material parameters are constant. The specific heat c of a material is the energy required to heat up a kilogram by one degree kelvin (in SI units, about which more shortly), so the thermal energy in the section of the rod in $a \leq x \leq a + h$ is given by

$$A \int_a^{a+h} \rho c T(x,t) \, \mathrm{d}x.$$

- We now introduce the heat flux $q(x,t)$ in the positive x-direction, which is the rate at which thermal energy is transported through a cross-section of the rod at station x at time t in the positive x-direction per unit cross-sectional area per unit time, *i.e.* the rate of flow of thermal energy along the rod. By definition, the rate at which thermal energy enters the section through its left-hand cross-section in the plane $x = a$ is $Aq(a,t)$. Similarly, the rate at which thermal energy leaves the section through the right-hand cross-section in the plane $x = a + h$ is $Aq(a+h,t)$. Hence, with our sign convention on the heat flux, the net rate at which thermal energy enters the section is

$$Aq(a,t) - Aq(a+h,t).$$

- Assuming insulated lateral surfaces and no external heating (*e.g.* due to microwave heating), conservation of energy states that the rate of change of the thermal energy in the section is equal to the net rate at which thermal energy enters the section, so that

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(A \int_a^{a+h} \rho c T(x,t) \, \mathrm{d}x \right)}_{(I)} = \underbrace{Aq(a,t)}_{(II)} - \underbrace{Aq(a+h,t)}_{(III)},$$

where we have labeled the three terms in order to summarize their physical significane as follows:

(I) is the time rate of change of thermal energy in the section in $a \le x \le a+h$;

(II) is the rate at which thermal energy enters the section through $x = a$;

(III) is the rate at which thermal energy leaves the section through $x = a + h$.

- We note this integral conservation law is also true for $h < 0$ with appropriate reinterpretation of the terms.

- Assuming T_t is continuous, Leibniz's Integral Rule with a and $a+h$ constant gives

$$\frac{\rho c}{h} \int_a^{a+h} T_t(x,t)\,\mathrm{d}x + \frac{q(a+h,t) - q(a,t)}{h} = 0,$$

where we have also rearranged into a form that will enable us to take the limit $h \to 0$.

- In particular, to take the limit $h \to 0$, we apply the Fundamental Theorem of Calculus to the first term (assuming T_t is continuous in a neighbourhood of a) and use the definition of the partial derivative of q with respect to x (assuming it to exist and to be continuous at a), to obtain the partial differential equation

$$\rho c T_t + q_x = 0, \qquad (\dagger)$$

which relates the time rate of change of the temperature and the spatial rate of change of the heat flux.

- To make further progress we must decide how the heat flux $q(x,t)$ depends on the temperature $T(x,t)$. This is called a constitutive relation and cannot be deduced, relying instead on some assumptions about the physical properties of the material under consideration. An example of a simple constitutive relation is Hooke's law of Prelims Dynamics for the extension of a spring — we note that

(i) a "thought-experiment" suggests this law is reasonable;

(ii) it could be confirmed experimentally;

(iii) it will almost certainly fail under "extreme" conditions.

- To close our model for heat conduction we will adopt **Fourier's Law**, which is is the constitutive law given by

$$q = -kT_x, \qquad (\ddagger)$$

where k is the thermal conductivity of the rod, which is another material parameter that we take to be constant.

- The minus sign in Fourier's law means that thermal energy flows down the temperature gradient, *i.e.* from high to low temperatures. Physical experiments confirm that this is an excellent approximation in many practical applications. We note that a good conductor of heat (such as silver) will have a higher thermal conductivity than a poor conductor of heat (such as glass).

- Substituting ($\ddagger$) into ($\dagger$), we arrive at the heat or diffusion equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2},$$

where the thermal diffusivity κ is defined by

$$\kappa = \frac{k}{\rho c}.$$

Units and nondimensionalisation

- **Notation:** We denote by $[p]$ the dimension of the quantity p in fundamental dimensions (M, L, T, Θ *etc*) or *e.g.* SI units (kg, m, s, K *etc*). We will work with the latter and recall that kelvin K is the SI unit of temperature, the newton N is the SI derived unit of force ($1\,\mathrm{N} = 1\,\mathrm{kg\,m\,s^{-1}}$), while the joule J is the SI derived unit of energy ($1\,\mathrm{J} = 1\,\mathrm{N\,m}$).

- Both sides of an equation modelling a physical process must have the same dimensions, *e.g.* in the integral conservation law above,

$$[(1)] = [(2)] = [(3)] = \mathrm{J\,s^{-1}},$$

while in the heat equation above,

$$[T_t] = [\kappa T_{xx}] = \mathrm{K\,s^{-1}}.$$

- We can exploit this fact to determine the dimensions of parameters and to check that solutions are dimensionally correct.

- For example, using Fourier's Law we find that the dimensions of the thermal conductivity are given by

$$[k] = \frac{[q]}{[T_x]} = \frac{\mathrm{J\,m^{-2}\,s^{-1}}}{\mathrm{K\,m^{-1}}} = \mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}},$$

and using the heat equation we find that the dimensions of the thermal diffusivity are given by

$$[\kappa] = \frac{[T_t]}{[T_{xx}]} = \frac{\mathrm{K\,s^{-1}}}{\mathrm{K\,m^{-2}}} = \mathrm{m^2\,s^{-1}}.$$

- We summarize in the table below the SI units of all of the variables and parameters involved in the derivation of the one-dimensional heat equation.

Symbol	Quantity	SI units
x	Axial distance	m
t	Time	s
T	Absolute temperature	K
q	Heat flux in positive x-direction	$\mathrm{J\,m^{-2}\,s^{-1}}$
A	Cross-sectional area	$\mathrm{m^2}$
ρ	Rod density	$\mathrm{kg\,m^{-3}}$
c	Rod specific heat	$\mathrm{J\,kg^{-1}\,K^{-1}}$
k	Rod thermal conductivity	$\mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}}$
κ	Rod thermal diffusivity	$\mathrm{m^2\,s^{-1}}$

- **Nondimensionalisation:** Method of scaling variables with typical values to derive dimensionless equations. These usually contain dimensionless parameters that characterise the relative importance of the physical mechanisms in the model. We illustrate the method with an example.

Example: heat conduction in a finite rod

- Consider the initial boundary value problem (IBVP) for the temperature $T(x, t)$ in a rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$

with the boundary conditions

$$T(0,t)=T_0, \quad T(L,t)=T_1 \quad \text{for} \quad t>0, \tag{2}$$

and the initial condition

$$T(x,0)=T_2\frac{x}{L}\left(1-\frac{x}{L}\right) \quad \text{for} \quad 0<x<L, \tag{3}$$

where T_0, T_1 snd T_2 are prescribed constant temperatures.

- There are five dimensional parameters, namely κ, L, T_0, T_1 and T_2.
- We nondimensionalise by scaling

$$x=L\widehat{x}, \quad t=\tau\widehat{t}, \quad T(x,t)=T_2\widehat{T}(\widehat{x},\widehat{t}),$$

where L, τ and T_2 are a typical lengthscale, timescale and temperature, respectively, so that the quantities $\widehat{x}$, $\widehat{t}$ and $\widehat{T}$ are dimensionless.

- By the chain rule,

$$\begin{aligned}\frac{\partial T}{\partial t}&=T_2\frac{\partial\widehat{T}}{\partial\widehat{t}}\frac{\mathrm{d}\widehat{t}}{\mathrm{d}t}=\frac{T_2}{\tau}\frac{\partial\widehat{T}}{\partial\widehat{t}},\\ \frac{\partial T}{\partial x}&=T_2\frac{\partial\widehat{T}}{\partial\widehat{x}}\frac{\mathrm{d}\widehat{x}}{\mathrm{d}x}=\frac{T_2}{L}\frac{\partial\widehat{T}}{\partial\widehat{x}}, \quad \textit{etc.}\end{aligned}$$

- Hence, the dimensional problem (1)-(3) for the dimensional temperature $T(x,t)$ implies that the corresponding dimensionless problem for the dimensionless temperature $\widehat{T}(\widehat{x},\widehat{t})$ is given by

$$\frac{\partial\widehat{T}}{\partial\widehat{t}}=D\frac{\partial^2\widehat{T}}{\partial\widehat{x}^2} \quad \text{for} \quad 0<\widehat{x}<1,\ \widehat{t}>0, \tag{1$'$}$$

with the boundary conditions

$$\widehat{T}(0,\widehat{t})=\alpha_0, \quad \widehat{T}(1,t)=\alpha_1 \quad \text{for} \quad \widehat{t}>0, \tag{2$'$}$$

and the initial condition

$$\widehat{T}(\widehat{x},0)=\widehat{x}(1-\widehat{x}) \quad \text{for} \quad 0<\widehat{x}<1, \tag{3$'$}$$

where the dimensionless parameters D, α_0 and α_1 are defined by

$$D=\frac{\kappa\tau}{L^2}, \quad \alpha_0=\frac{T_0}{T_2}, \quad \alpha_1=\frac{T_1}{T_2}.$$

- We can further reduce the number of dimensionless parameters by choosing the timescale τ so that $D=1$, *i.e.* by choosing $\tau=L^2/\kappa$, which is the timescale for conductive transport of heat over a distance L because it balances both terms in (1$'$). With this choice of timescale, we note that if $\widehat{T}=\widehat{T}(\widehat{x},\widehat{t};\alpha_0,\alpha_1)$ is a solution of (1$'$)-(3$'$), then a solution T of (1)-(3) is given by

$$\frac{T}{T_2}=\widehat{T}\left(\frac{x}{L},\frac{\kappa t}{L^2};\frac{T_0}{T_2},\frac{T_1}{T_2}\right).$$

i.e. T/T_2 must be a function of x/L and $\kappa t/L^2$. This means that we can compare heat problems on different scales: for example, two systems with different L and κ will exhibit comparable behaviour on the same time scales if L^2/κ is the same in each problem.

Heat conduction in a finite rod

- Consider the initial boundary value problem (IBVP) for the temperature $T(x,t)$ in a rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$

with the boundary conditions

$$T(0,t) = 0, \quad T(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2}$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3}$$

where the initial temperature profile $f(x)$ is given.

- We will construct a solution using **Fourier's method**, which consists of the following three steps:

  (I) Use the method of separation of variables to find the countably infinite set of nontrivial separable solutions satisfying the partial differential equation (1) and boundary conditions (2), each containing an arbitrary constant.

  (II) Use the principle of superposition — that the sum of any number of solutions of a linear problem is also a solution (assuming convergence) — to form the general series solution that is the infinite sum of the separable solutions of the partial differential equation and boundary conditions.

  (III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the initial condition (3).

Remarks

- (1) and (2) are linear since, if T_1 and T_2 satisfy them, then so does $\alpha T_1 + \beta T_2$ for all $\alpha,\ \beta \in \mathbb{R}$.

- To verify that the resulting series is actually a solution of the partial differential equations, we need it to converge sufficiently rapidly that T_t and T_{xx} can be computed by termwise differentiation — we largely gloss over such issues.

Step I

- We begin by seeking a nontrivial separable solution of the form $T = F(x)G(t)$ for which the partial differential equation (1) gives $F(x)G'(t) = \kappa F''(x)G(t)$, with a prime $'$ denoting the derivative with respect to the argument.

- Separating the variables by assuming $F(x)G(t) \neq 0$ therefore gives

$$\frac{F''(x)}{F(x)} = \frac{G'(t)}{\kappa G(t)}.$$

The left-hand side of this expression is independent of t, while the right-hand side independent of x. Since the left-hand side is equal to the right-hand side, they must both be independent of x and t, *i.e.* LHS = RHS = $-\lambda$ for some constant $\lambda \in \mathbb{R}$.

- The boundary condition at $x = 0$ in (2) implies that $F(0)G(t) = 0$ for $t > 0$. Since we're seeking solutions T that are nontrivial (*i.e.* not identically equal to zero), there must exist a time $t > 0$ such that $G(t) \neq 0$, and hence we must impose on $F(x)$ the boundary condition $F(0) = 0$. Similarly, the boundary condition at $x = L$ in (2) implies that $F(L) = 0$.

- In summary, we have deduced that $F(x)$ satisfies the boundary value problem given by the ordinary differential equation

$$-F''(x) = \lambda F(x) \quad \text{for} \quad 0 < x < L, \tag{†}$$

with the boundary conditions

$$F(0) = 0, \quad F(L) = 0, \tag{‡}$$

where $\lambda \in \mathbb{R}$

- Now need to find all $\lambda \in \mathbb{R}$ such that the boundary value problem (†)-(‡) for $F(x)$ has a nontrivial solution.

- There are three cases to consider, as follows:

  (i) $\underline{\lambda = -\omega^2 \ (\omega > 0 \text{ wlog})}$

  $$(†) \implies F'' - \omega^2 F = 0 \implies F(x) = A\cosh(\omega x) + B\sinh(\omega x), \text{ where } A, B \in \mathbb{R}.$$

  $$(‡) \implies A = 0,\ B\sinh(\omega L) = 0 \implies F = 0.$$

  (ii) $\underline{\lambda = 0}$

  $$(†) \implies F'' = 0 \implies F(x) = A + Bx, \text{ where } A, B \in \mathbb{R}.$$

  $$(‡) \implies A = 0,\ BL = 0 \implies F = 0.$$

  (iii) $\underline{\lambda = \omega^2 \ (\omega > 0 \text{ wlog})}$

  $$(†) \implies F'' + \omega^2 F = 0 \implies F(x) = A\cos(\omega x) + B\sin(\omega x), \text{ where } A, B \in \mathbb{R}.$$

  $$(‡) \implies A = 0,\ B\sin(\omega L) = 0.$$

  $$B \neq 0 \implies \sin\omega L = 0 \implies \omega L = n\pi,\ n \in \mathbb{N} \setminus \{0\}.$$

- Hence, the nontrivial solutions of the partial differential equation (†)-(‡) are given by

$$F(x) = B\sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad \lambda = \frac{n^2\pi^2}{L^2}, \quad n \in \mathbb{N} \setminus \{0\}.$$

- Since $G(t)$ satisfies the ordinary differential equation $G' = -\lambda\kappa G$, we deduce that

$$G(t) = C\exp\left(-\lambda\kappa t\right),$$

where $C \in \mathbb{R}$.

- Since $T(x,t) = F(x)G(t)$, we conclude that the nontrivial separable solutions of the heat equation (1) that satisfy the boundary conditions (2) are given by

$$T_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

where n is a positive integer, b_n is a constant (equal to BC in the analysis above) and we have introduced the subscript n on T_n and b_n to enumerate the countably infinite set of such solutions.

Step II

- Since (1)-(2) are linear, a formal application of the principle of superposition implies that the general series solution is given by

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right).$$

Step III

- The initial condition (3) can only be satisfied by the general series solution if

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad 0 < x < L,$$

  so that we need to find the Fourier sine series for f.

- The theory of Fourier series implies that the Fourier coefficients b_n are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad \text{for} \quad n \in \mathbb{N}\setminus\{0\}, \tag{‡‡}$$

- Hence, we have derived a solution in the form of an infinite trigonometric series.

Remarks

- The integral expressions for the Fourier coefficients in (‡‡) may be derived via the orthogonality relations

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{L}{2}\delta_{mn} \qquad (m,\, n \in \mathbb{N}\setminus\{0\})$$

  by assuming that the orders of summation and integration may be interchanged, as follows:

$$\begin{aligned}
\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x &= \frac{2}{L}\int_0^L \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \sum_{m=1}^{\infty} b_m \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \sum_{m=1}^{\infty} b_m \delta_{mn} \\
&= b_n \qquad \text{for} \quad n \in \mathbb{N}\setminus\{0\}.
\end{aligned}$$

- If f and f' are piecewise continuous on $(0,L)$, then the Fourier Convergence Theorem implies that the Fourier sine series for f converges to $(f(x_+) + f(x_-))/2$ for $x \in (0,L)$ and to 0 for $x = 0, L$. Thus, Fourier's method can even handle jump discontinuities in the initial temperature profile, with the caveat that the truncated series solution would exhibit Gibb's phenomenon at time $t = 0$.

- It can be rigorously proven (using methods from Prelims Analysis) that for such initial conditions, the series solution converges and is a solution of the initial boundary value problem (1)–(3). Since $T_n(x,t)$ decays exponentially as $n \to \infty$ for fixed $x \in (0, L)$ and fixed $t > 0$, it may also be shown that all partial derivatives with respect to x and t exist and may be derived by termwise partial differentiation of the series!

- There are several important implications of the last two remarks:

  - the heat equation *smoothes* out instantaneously even irregular initial temperature profiles;
  - as soon as $t > 0$, most of the high frequency terms $T_n(x,t)$ for $n \gg 1$ will be extremely small, so that the solution may be well approximated by only a handful of terms;
  - the temperature tends to zero exponentially quickly as $\kappa t/L^2 \to \infty$, *i.e.* on the timescale of heat conduction, with the thermal energy initially stored in the rod being conducted out of the ends of the rod on this timescale.

- Consider the initial profile given by

$$f(x) = \begin{cases} T^* & \text{for } L_1 < x < L_2\,, \\ 0 & \text{otherwise,} \end{cases}$$

where T^*, L_1 and L_2 are constants, for which the Fourier coefficients are given by

$$b_n = \frac{2}{L}\int_{L_1}^{L_2} T^* \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2T^*}{n\pi}\left(\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right)$$

for positive integers n. We plot below snapshots of the partial sums of the truncated series solution with 100 terms for $L_1/L = 0.2$, $L_2/L = 0.4$ and $100\kappa t/L^2 = 0$, 0.25, 0.5, 1, 2, 4, 8, 16 and 32, which illustrates all of the main features of the solution discussed above.