Fourier Series & PDEs: Lectures 7-8

Uniqueness

- In the last lecture we considered the initial boundary value problem for the temperature $T(x,t)$ given by the heat equation
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$
with the so-called homogeneous Dirichlet boundary conditions
$$T(0,t) = 0, \quad T(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2}$$
and the initial condition
$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3}$$
where the initial temperature profile $f(x)$ is given.
- We used Fourier's method to construct an infinite series solution, but is it the only solution?
- **Uniqueness Theorem:** The initial boundary value problem (1)–(3) has only one solution.
- **Proof of the uniqueness theorem:** Our strategy is to show that the difference between any two solutions much vanish.
- Thus we suppose that $T(x,t)$ and $\widetilde{T}(x,t)$ are solutions to (1)–(3) and let
$$W(x,t) = T(x,t) - \widetilde{T}(x,t)$$
be their difference.
- By linearity, (1)-(3) imply that $W(x,t)$ satisfies the heat equation
$$\frac{\partial W}{\partial t} = \frac{\partial T}{\partial t} - \frac{\partial \widetilde{T}}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} - \kappa \frac{\partial^2 \widetilde{T}}{\partial x^2} = \kappa \frac{\partial^2 W}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1'}$$
with the boundary conditions
$$W(0,t) = T(0,t) - \widetilde{T}(0,t) = 0, \quad W(L,t) = T(L,t) - \widetilde{T}(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2'}$$
and the initial condition
$$W(x,0) = T(x,0) - \widetilde{T}(x,0) = f(x) - f(x) = 0 \quad \text{for} \quad 0 < x < L. \tag{3'}$$
- Since W is the temperature in a metal rod whose initial temperature is everywhere zero and whose ends are held at zero temperature thereafter, on physical grounds we expect the rod to remain at zero temperature, *i.e.* $W = 0$ for $0 \le x \le L$ and $t \ge 0$, which is what we need to show to prove uniqueness.
- The trick is to analyse the integral $I(t)$ defined by
$$I(t) := \frac{1}{2} \int_0^L W(x,t)^2 \, \mathrm{d}x.$$
- Evidently $I(t) \ge 0$ for $t \ge 0$ and $I(0) = 0$ by (3′).

- But, for $t > 0$,

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}t} &= \int_0^L W \frac{\partial W}{\partial t}\,\mathrm{d}x && \text{(by Liebniz Integral Rule)}\\
&= \int_0^L W\kappa \frac{\partial^2 W}{\partial x^2}\,\mathrm{d}x && \text{(by (1'))}\\
&= \left[\kappa W \frac{\partial W}{\partial x}\right]_{x=0}^{x=L} - \kappa \int_0^L \frac{\partial W}{\partial x}\frac{\partial W}{\partial x}\,\mathrm{d}x && \text{(by integration by parts)}\\
&= -\kappa \int_0^L \left(\frac{\partial W}{\partial x}\right)^2 \mathrm{d}x && \text{(by (2'))}\\
&\leq 0
\end{aligned}$$

  which means that $I(t)$ cannot increase, so that $I(t) \leq I(0) = 0$ for $t \geq 0$.

- Since $I(t) \geq 0$ and $I(t) \leq 0$ for $t \geq 0$, we deduce that $I(t) = 0$ for $t \geq 0$, and hence that $W(x,t) = 0$ for $0 \leq x \leq L$, $t \geq 0$ (assuming continuity of W there), which completes the proof.

- We note that this method of proof also works for the so-called homogeneous Neumann boundary conditions (about which more shortly) given by

$$\frac{\partial T}{\partial x}(0,t) = 0, \quad \frac{\partial T}{\partial x}(L,t) = 0 \quad \text{for} \quad t > 0,$$

  as well as for the so-called homogeneous Robin boundary conditions of the form

$$\frac{\partial T}{\partial x}((0,t) = -\alpha T(0,t), \quad T_x(L,t) = \alpha T(L,t) \quad \text{for} \quad t > 0,$$

  where α is a positive parameter, since in both cases it may be shown that

$$\left[\kappa W \frac{\partial W}{\partial x}\right]_{x=0}^{x=L} \leq 0.$$

Example: inhomogeneous Dirichlet boundary conditions

- Consider the initial boundary value problem for the temperature $T(x,t)$ given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$

  with the inhomogeneous Dirichlet boundary conditions

$$T(0,t) = T_L, \quad T(L,t) = T_R \quad \text{for} \quad t > 0, \tag{2}$$

  and the initial condition

$$T(x,0) = 0 \quad \text{for} \quad 0 < x < L, \tag{3}$$

  where T_L and T_R are prescribed constant temperatures, not both zero.

- Lets apply Fourier's method. In step I we need to find the nontrivial separable solutions $T(x,t) = F(x)G(t)$ of the heat equation (1) and boundary conditions (2). But the latter would require

$$F(0)G(t) = T_L, \quad F(L)G(t) = T_R \quad \text{for} \quad t > 0,$$

forcing G to be constant. It follows that the only nontrivial separable solution of (1)–(2) is the time-independent or steady-state solution (about which more shortly). Since this cannot satisfy the initial condition (3), Fourier's method fails because the boundary conditions (2) are not homogeneous.

- However, we can transform the problem into one amenable to Fourier's method, as follows.
- On physical grounds, we conjecture that $T(x,t) \to S(x)$ as $t \to \infty$, where $S(x)$ is the aforementioned steady-state solution of (1)–(2), which satisfies

$$0 = \kappa \frac{\mathrm{d}^2 S}{\mathrm{d}x^2} \quad \text{for} \quad 0 < x < L,$$

with $S(0) = T_L$ and $S(L) = T_R$. Thus, $S(x)$ has the linear temperature profile given by

$$S(x) = T_L\left(1 - \frac{x}{L}\right) + T_R\left(\frac{x}{L}\right);$$

we note that in steady state thermal energy is conducted steadily along the rod with constant heat flux

$$q = -k\frac{\partial T}{\partial x} = \frac{k(T_L - T_R)}{L},$$

so that heat flows steadily in the positive x-direction for $T_L > T_R$.

- We now observe that if we let

$$T(x,t) = S(x) + U(x,t),$$

then by linearity (1)-(3) imply that $U(x,t)$ satisfies the initial boundary value problem given by the heat equation

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1'}$$

with the homogeneous Dirichlet boundary conditions

$$U(0,t) = 0, \quad U(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2'}$$

and the initial condition

$$U(x,0) = -S(x) = -T_L\left(1 - \frac{x}{L}\right) - T_R\left(\frac{x}{L}\right) \quad \text{for} \quad 0 < x < L. \tag{3'}$$

- The initial boundary value problem (1′)–(3′) for $U(x,t)$ is amenable to Fourier's method: we solved it last lecture to find the solution given by

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

where

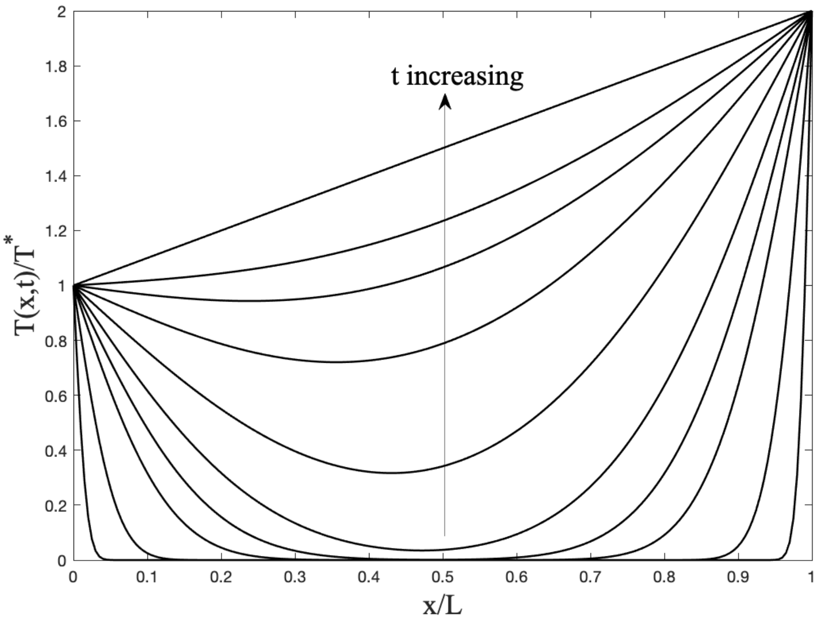
$$b_n = -\frac{2}{L}\int_0^L S(x)\sin\left(\frac{n\pi x}{L}\right) = -\frac{2}{n\pi}(T_L - (-1)^n T_R).$$

- **Remarks:**

  (1) We note that the parameters T_L and T_R in the boundary conditions (2) ended up in the initial condition (3′) — hence the method above is sometimes called "the method of shifting the data".

  (2) Since $U(x,t) \to 0$ as $\kappa t/L^2 \to \infty$, we can verify our conjecture that $T(x,t) \to S(x)$ as $\kappa t/L^2 \to \infty$.

(3) We plot below snapshots of the partial sums of the truncated series solution with 100 terms for $T_L = T^*$, $T_R = 2T^*$ and $100\kappa t/L^2 = 0.01$, 0.1, 0.5, 1, 2, 5, 10, 15, 20 and 100. The profiles illustrate the manner in which heat conduction rapidly drives the temperature toward the linear steady-state temperature profile.



Example: Neumann boundary conditions

- Consider the initial boundary value problem for the temperature $T(x,t)$ given by the heat equation
$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$
with the so-called homogeneous Neumann boundary conditions
$$\frac{\partial T}{\partial x}(0,t) = 0, \quad \frac{\partial T}{\partial x}(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2}$$
and the initial condition
$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L. \tag{3}$$

- We note that that the ends of the rod are thermally insulated because the heat flux
$$q = -k\frac{\partial T}{\partial x} = 0 \quad \text{at} \quad x = 0 \text{ and } x = L \text{ for } t > 0$$
by Fourier's law and the boundary conditions (2).

- Fourier's method is applied on problem sheet 4 to show that the solution is given by
$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$
where the constants a_n are the Fourier coefficients of the Fourier cosine series for f given by
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

- **Remarks:**

  (1) The constant (separable) solution $T = a_0/2$ of (1)-(2) comes from the case in which the separation constant is zero.

  (2) The temperature $T(x,t) \to a_0/2$ as $\kappa t/L^2 \to \infty$, *i.e.* the temperature tends to the mean of the initial temperature. Thus, the rod retains all of its its initial thermal energy because all of its surfaces are insulated and heat conduction causes the temperature to approach on the timescale of heat conduction the steady state solution in which T is spatially uniform.

Example: inhomogeneous heat equation and boundary conditions

- Consider the IBVP for the temperature $T(x,t)$ in a rod of length L given by the inhomogeneous heat equation
$$\rho c \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$
with the inhomogeneous Neumann boundary conditions
$$T_x(0,t) = \phi(t), \quad T_x(L,t) = \psi(t) \quad \text{for} \quad t > 0, \tag{2}$$
and the initial condition
$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3}$$
where the functions $Q(x,t)$, $\phi(t)$, ψ(t) and $f(x)$ are given.

- We note that Q is the volumetric heat source (*e.g.* due to radiation or chemical reactions) and the heat flux in the positive direction $q = -kT_x$ according to Fourier's law, so that the boundary conditions prescribe q at each end of the rod.

- In general Fourier's method cannot be used to solve the IBVP for T because the heat equation and boundary conditions are inhomogeneous (*i.e.* Q, ϕ and ψ are non-zero). We now describe a generalization of Fourier's method that works.

- We deal first with the boundary conditions: if we let $T(x,t) = S(x,t) + U(x,t)$, where
$$S(x,t) = -\phi(t)\frac{(x-L)^2}{2L} + \psi(t)\frac{x^2}{2L},$$
say, is chosen to satisfy the boundary conditions (2), then by linearity the initial boundary value problem (1)-(3) for $T(x,t)$ implies that the initial boundary value problem for $U(x,t)$ is given by
$$\rho c \frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \widetilde{Q}(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1$'$}$$
with the inhomogeneous boundary conditions
$$U_x(0,t) = 0, \quad U_x(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2$'$}$$
and the initial condition
$$T(x,0) = \widetilde{f}(x) \quad \text{for} \quad 0 < x < L, \tag{3$'$}$$
where the functions
$$\widetilde{Q}(x,t) = Q(x,t) + k\frac{\partial^2 S}{\partial x^2} - \rho c\frac{\partial S}{\partial t}, \quad \widetilde{f}(x) = f(x) - S(x,0)$$
are known in terms of $Q(x,t)$, $\phi(t)$, $\psi(t)$ and $f(x)$.

- Thus, the boundary conditions have been rendered homogeneous by 'shifting the data' in the sense that both $\phi(t)$ and $\psi(t)$ have moved from the boundary conditions (2) for $T(x,t)$ into the heat equation ($1'$) and initial conditions ($2'$) for $U(x,t)$.

- If $\widetilde{Q} \equiv 0$, then we can solve the initial boundary value problem for $U(x,t)$ using Fourier's method as outlined above to obtain

$$U(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2 kt}{\rho c L^2}\right),$$

where the Fourier coefficients a_n are chosen to satisfy the initial condition so that

$$a_n = \frac{2}{L}\int_0^L \widetilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

- This series solution for $U(x,t)$ suggests that if $\widetilde{Q}$ is not identically zero, then we should seek a solution for U in the form of the Fourier cosine series

$$U(x,t) = \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t)\cos\left(\frac{n\pi x}{L}\right),$$

where the Fourier coefficients $U_n(t)$ depend on time and are to be determined. From the formulae for the Fourier coefficients of a Fourier cosine series, we deduce that $U_n(t)$ are given in terms of $U(x,t)$ by the integral expressions

$$U_n(t) = \frac{2}{L}\int_0^L U(x,t)\cos\left(\frac{n\pi x}{L}\right)\,\mathrm{d}x.$$

- By Leibniz's integral rule,

$$\begin{aligned}
\rho c\frac{\mathrm{d}U_n}{\mathrm{d}t} &= \frac{2}{L}\int_0^L \rho c\frac{\partial U}{\partial t}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2}{L}\int_0^L \left(k\frac{\partial^2 U}{\partial x^2} + \widetilde{Q}\right)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \frac{2k}{L}\int_0^L \frac{\partial^2 U}{\partial x^2}(x,t)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x + \widetilde{Q}_n(t),
\end{aligned}$$

where in the second equality we used the heat equation ($1'$) and in the last equality we introduced the functions defined by

$$\widetilde{Q}_n(t) = \frac{2}{L}\int_0^L \widetilde{Q}(x,t)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x,$$

which are the coefficients of the Fourier cosine series for $\widetilde{Q}(x,t)$.

- <u>Question:</u> How do we deal with the U_{xx} integral?

- Answer: By two integration by parts using the boundary conditions (2′). This may be accomplished in one step by using the identity

$$(uv' - u'v)' = uv'' - u''v \implies \big[uv' - u'v\big]_0^L = \int_0^L uv'' - u''v \,\mathrm{d}x,$$

with $u = U$ and $v = \cos(n\pi x/L)$, giving

$$\underbrace{\left[U\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) - U_x \cos\left(\frac{n\pi x}{L}\right)\right]_0^L}_{=0 \text{ by } (2')} = \int\limits_0^L U\left(-\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) - U_{xx}\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x,$$

so that

$$\frac{2}{L}\int\limits_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}\frac{2}{L}\int\limits_0^L U\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}U_n\,.$$

- Combining the equations above, we find that $U_n(t)$ is governed by the ordinary differential equation

$$\rho c \frac{\mathrm{d}U_n}{\mathrm{d}t} + \frac{kn^2\pi^2}{L^2}U_n = \widetilde{Q}_n(t) \quad \text{for} \quad t > 0,$$

with the initial condition (3′) for $U(x,t)$ giving the initial condition

$$U_n(0) = \frac{2}{L}\int_0^L \widetilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

Remarks

(1) We have reduced the problem to a countably infinite set of initial value problems for $U_n(t)$, each of which may be solved using an integrating factor (as in Prelims Introductory Calculus). We note that if $\widetilde{Q}(x,t) \equiv 0$, then $\widetilde{Q}_n(t) \equiv 0$ and we recover the solution above for $U_n(t)$ obtained by Fourier's method.

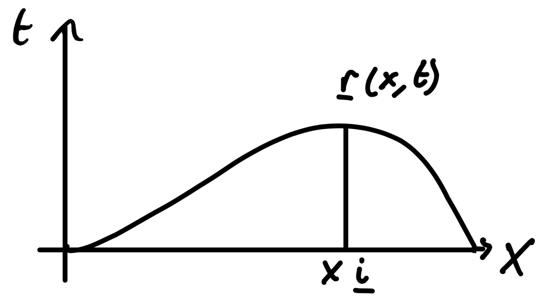
(2) We chose the function $S(x,t)$ so that it satisfied the inhomogeneous boundary conditions (2). The choice is not unique. The smoother the $2L$-periodic even extension for $S(x,t)$, the faster the decay of the Fourier coefficients $U_n(t)$ as $n \to \infty$ for fixed $t \geq 0$. There is therefore a trade off between the rate of convergence of the series solution for $U(x,t)$ and the ease of computation of the Fourier coefficients $\widetilde{Q}_n(t)$, the initial values $U_n(0)$ and hence the solution for $U_n(t)$.

The wave equation

Derivation in one dimension

- Consider the small transverse vibrations of a homogeneous extensible elastic string stretched initially along the x-axis at time $t = 0$.

- A point at $x\boldsymbol{i}$ at time $t = 0$ is displaced to $\boldsymbol{r}(x,t) = x\boldsymbol{i} + y(x,t)\boldsymbol{j}$ at time $t > 0$, where the transverse displacement $y(x,t)$ is to be determined. We illustrate the geometrical setup in the schematic below.



- Consider the section of string in the fixed region $a \le x \le a+h$, where a and h are arbitrary constants.

- The linear momentum of the section of the string in $a \le x \le a+h$ is

$$\int_a^{a+h} \rho \frac{\partial \boldsymbol{r}}{\partial t}\, \mathrm{d}x,$$

where ρ is the constant line density of the string (with $[\rho] = \mathrm{kg\,m^{-1}}$).

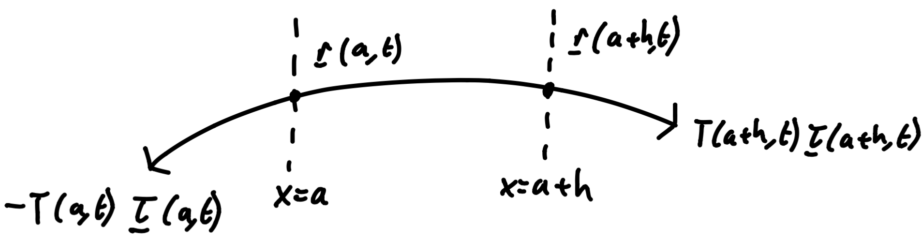
- The string offers no resistance to bending (*cf.* a ruler) in the sense that the string to the right of the point $\boldsymbol{r}(x,t)$ exerts at that point a tangential force $T(x,t)\boldsymbol{\tau}(x,t)$ on the string to the left, where $T(x,t)$ is the tension ($[T] = \mathrm{N} = \mathrm{kg\,m\,s^{-2}}$) and $\boldsymbol{\tau} = \boldsymbol{r}_x / |\boldsymbol{r}_x|$ is the unit tangent vector pointing in the positive x-direction. Note that Newton's third law implies that the string to the left of the point $\boldsymbol{r}(x,t)$ exerts at that point a tangential force $-T(x,t)\boldsymbol{\tau}(x,t)$ on the string to the right.

- Assuming the tension is so large that the effects of gravity and air resistance may be neglected, the forces acting on the ends of the section of string in $a \le x \le a+h$ are

  (i) the force $T(a+h,t)\boldsymbol{\tau}(a+h,t)$ exerted at the right-hand end at $\boldsymbol{r}(a+h,t)$ by the string to the right of the section;

  (ii) the force $-T(a,t)\boldsymbol{\tau}(a,t)$ exerted at the left-hand end at $\boldsymbol{r}(a,t)$ by the string to the left of the section.

  We illustrate the forces and where they act on the section in the schematic below.



- We are now in a position to apply Newton's Second Law, which states that the rate of change of the linear momentum of the section of string in $a \le x \le a+h$ is equal to the net force acting on its ends, so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{a+h} \rho \frac{\partial \boldsymbol{r}}{\partial t}\, \mathrm{d}x \right) = T(a+h,t)\boldsymbol{\tau}(a+h,t) - T(a,t)\boldsymbol{\tau}(a,t).$$

- Assuming $\boldsymbol{r}_{tt}$ is continuous, Leibniz's Integral Rule with a and $a+h$ constant gives

$$\frac{1}{h}\int_a^{a+h} \rho \frac{\partial^2 \boldsymbol{r}}{\partial t^2}\, \mathrm{d}x = \frac{T(a+h,t)\boldsymbol{\tau}(a+h,t) - T(a,t)\boldsymbol{\tau}(a,t)}{h},$$

where we divided by h in anticipation of taking the limit $h \to 0$.

- In particular, to take the limit $h \to 0$, we apply the Fundamental Theorem of Calculus to the first term (assuming $\boldsymbol{r}_{tt}$ is continuous in a neighbourhood of a) and use the definition of the partial derivative of $T\boldsymbol{\tau}$ with respect to x (assuming it to exist and to be continuous at a) to obtain
$$\rho \frac{\partial^2 \boldsymbol{r}}{\partial t^2} = \frac{\partial}{\partial x}\big(T\boldsymbol{\tau}\big).$$

- Recalling the definitions of $\boldsymbol{r}$ and $\boldsymbol{\tau}$, it follows that
$$\rho \frac{\partial^2 y}{\partial t^2}\boldsymbol{j} = \frac{\partial}{\partial x}\left(\frac{T\boldsymbol{i} + Ty_x\boldsymbol{j}}{(1+y_x^2)^{1/2}}\right).$$

- But we are also assuming that the transverse displacement is small in the sense that the slope of the string is small, *i.e.* $|y_x| \ll 1$.

- Since a Taylor expansion gives
$$\big(1+y_x^2\big)^{1/2} = 1 + \frac{1}{2}(y_x)^2 + \cdots \quad \text{for} \quad |y_x| \ll 1,$$
to a first approximation, *i.e.* neglecting quadratic and higher order terms,
$$\rho \frac{\partial^2 y}{\partial t^2}\boldsymbol{j} = \frac{\partial}{\partial x}\big(T\boldsymbol{i} + Ty_x\boldsymbol{j}\big). \tag{$\star$}$$

- The x-component of $(\star)$ implies that the tension T is spatially uniform, but could vary with time t, *e.g.* as when tuning a guitar string. We shall take the tension T to be constant, which is the case in many practical applications.

- The y-component of $(\star)$ then implies that
$$\rho \frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2},$$
giving the wave equation
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$
where $c = \sqrt{T/\rho}$ is the wave speed (for reasons that will become apparent).