Fourier Series & PDEs: Lectures 9-10

Units and nondimensionalisation

- Last lecture we showed that the small transverse displacement $y(x,t)$ of an elastic string is governed by wave equation
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$
where $c > 0$ is the constant wave speed.

- Consider the units of the variables (x, t and y) and parameter (c) in the wave equation. Since
$$[y_{tt}] = \mathrm{m\,s^{-2}}, \quad [y_{xx}] = \mathrm{m\,m^{-2}},$$
it follows that
$$\left[c^2\right] = \frac{[y_{tt}]}{[y_{xx}]} = \mathrm{m^2\,s^{-2}},$$
so that $[c] = \mathrm{m\,s^{-1}}$, *i.e.* c has the units of speed.

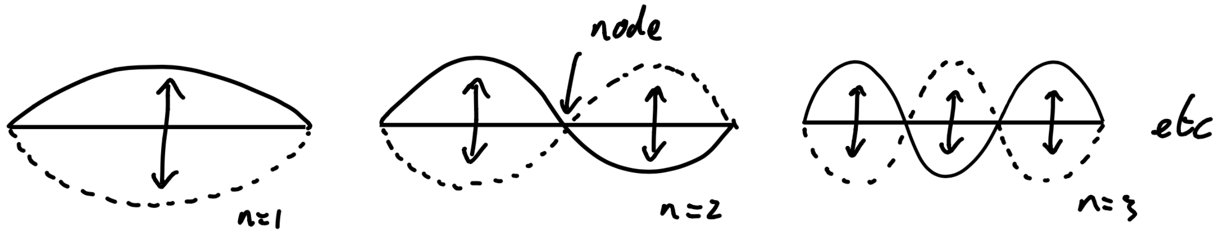
- Question: On what timescale does a displacement travel a distance L?

- Answer: If we nondimensionalize by scaling $x = L\hat{x}$, $t = t_0\hat{t}$, $y = H\hat{y}(\hat{x},\hat{t})$, then the wave equation becomes
$$\frac{H}{t_0^2}\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{Hc^2}{L^2}\frac{\partial^2 \hat{y}}{\partial \hat{x}^2};$$
the terms balance giving
$$\frac{\partial^2 \hat{y}}{\partial \hat{t}^2} = \frac{\partial^2 \hat{y}}{\partial \hat{x}^2}$$
provided $t_0 = L/c$, which is therefore the timescale for a displacement to travel a distance L.

Normal modes of vibration for a finite string

- Suppose an elastic string is stretched between $x = 0$ and $x = L$ and the ends held fixed, so that the small transverse displacement $y(x,t)$ of the string is governed by the wave equation
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L, \tag{1}$$
with the boundary conditions
$$y(0,t) = 0 \quad y(L,t) = 0. \tag{2}$$

- An experiment with a slinky suggests there exist discrete modes of vibration, as illustrated in the schematic below.



- To analyse mathematically the possible modes of vibration, we seek nontrivial separable solutions of the form $y = F(x)G(t)$.

- Substituting this expression into the heat equation (1) gives $F(x)G''(t) = c^2F''(x)G(t)$, so we may separate the variables for $FG \neq 0$ to obtain

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2G(t)}.$$

- The left-hand side of this expression is independent of t, while the right-hand side independent of x. Since the left-hand side is equal to the right-hand side, they must both be independent of x and t, *i.e.* LHS = RHS = $-\lambda$ for some constant $\lambda \in \mathbb{R}$.

- The boundary condition at $x = 0$ in (2) implies that $F(0)G(t) = 0$ for $t > 0$. Since we're seeking solutions y that are nontrivial, there must exist a time $t > 0$ such that $G(t) \neq 0$, and hence we must impose on $F(x)$ the boundary condition $F(0) = 0$. Similarly, the boundary condition at $x = L$ in (2) implies that $F(L) = 0$.

- In summary, we have deduced that $F(x)$ satisfies the boundary value problem given by the ordinary differential equation

$$-F''(x) = \lambda F(x) \quad \text{for} \quad 0 < x < L, \tag{$\dagger$}$$

with the boundary conditions

$$F(0) = 0, \quad F(L) = 0, \tag{$\ddagger$}$$

where $\lambda \in \mathbb{R}$

- We solved this problem in Lecture 6: the nontrivial solutions are given for positive integers n by

$$F(x) = B\sin\left(\frac{n\pi x}{L}\right) \quad \text{for} \quad \lambda = \left(\frac{n\pi}{L}\right)^2,$$

where B is an arbitrary constant; since $G'' + \lambda c^2 G = 0$, the corresponding solution for $G(t)$ is given by

$$G(t) = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right),$$

where C and D are arbitrary constants.

- Since $T(x,t) = F(x)G(t)$, we conclude that the nontrivial separable solutions or the normal models of (1)–(2) are given for positive integers n by

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right)$$

where a_n and b_n are arbitrary constants (with $a_n = BC$ and $b_n = BD$) and we have introduced the subscript n to enumerate the countably infinite set of such solutions.

Remarks

(1) The normal mode $y_n(x,t)$ is periodic in t with prime period

$$p = \frac{2\pi}{n\pi c/L} = \frac{2L}{nc}$$

and frequency (or pitch)

$$\frac{1}{p} = \frac{nc}{2L}.$$

(2) The first normal mode y_1 is called the fundamental mode, with associated fundamental frequency $\frac{c}{2L}$. All of the other modes have a frequency that is an integer multiple of the fundamental frequency.

(3) The predictions are consistent with the slinky experiment.

(4) The normal modes are an example of a standing wave because y_n is equal to a function of x multiplied by an oscillatory function of time.

Initial boundary value problem for a finite string

- Consider the initial boundary value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1}$$

with the Dirichlet boundary conditions

$$y(0,t) = 0, \quad y(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2}$$

and the initial conditions

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad 0 < x < L, \tag{3}$$

where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.

- We note that we impose two boundary conditions because the wave equation is second order in space (due to the y_{xx} term) and two initial conditions because the wave equation is second order in time (due to the y_{tt} term). In contract, for the heat equation $T_t = \kappa T_{xx}$, while we impose two boundary conditions because the heat equation is second order in space (due to the T_{xx} term), we impose only one initial condition because the heat equation is first order in time (due to the T_t term).

- We will use Fourier's method to find a series solution.

Step I: Find all nontrivial separable solutions of (1)-(2)

- We found above that these are the normal modes given by

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right),$$

where a_n, $b_n \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$.

Step II: Apply the principle of superposition

- Since (1)–(2) are linear, we can superimpose the normal modes (assuming convergence) to obtain the general series solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right). \tag{$\star$}$$

Step III: Use the theory of Fourier series to satisfy the initial conditions

- The initial conditions (3) can only be satisfied by ($\star$) if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L\,,$$

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L\,.$$

- Hence, a_n is the nth Fourier coefficient of the Fourier sine series for f, while $n\pi c b_n/L$ is the nth Fourier coefficient of the Fourier sine series for g, *i.e.*

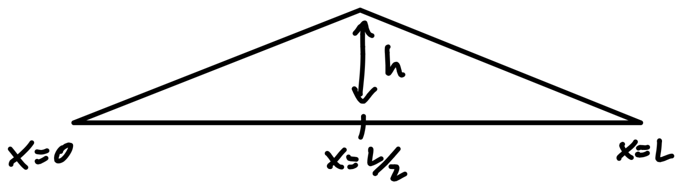
$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad \frac{n\pi c}{L}b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x. \qquad (\star\star)$$

Example: guitar string

- Suppose

$$f(x) = \begin{cases} 2hx/L & \text{for } 0 \le x \le L/2, \\ 2h(L-x)/L & \text{for } L/2 \le x \le L, \end{cases} \qquad g(x) = 0,$$

where h is a constant; we plot the graph of f below for $h > 0$.



- By $(\star\star)$,

$$a_n = \frac{2}{L}\int_0^{L/2} \frac{2hx}{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x + \frac{2}{L}\int_{L/2}^{L} \frac{2h(L-x)}{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right),$$

while $b_n = 0$.

- Since

$$\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{for } n = 2m, m \in \mathbb{N}\setminus\{0\}, \\ (-1)^m & \text{for } n = 2m+1, m \in \mathbb{N}, \end{cases}$$

it follows from $(\star)$ that a series solution is given by

$$y(x,t) = \frac{8h}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi ct}{L}\right).$$

Example: piano string

- Suppose

$$f(x) = 0, \quad g(x) = \begin{cases} v & \text{for } L_1 \le L \le L_2, \\ 0 & \text{otherwise.} \end{cases}$$

where v, L_1 and L_2 are constants.

- By $(\star\star)$, $a_n = 0$ and

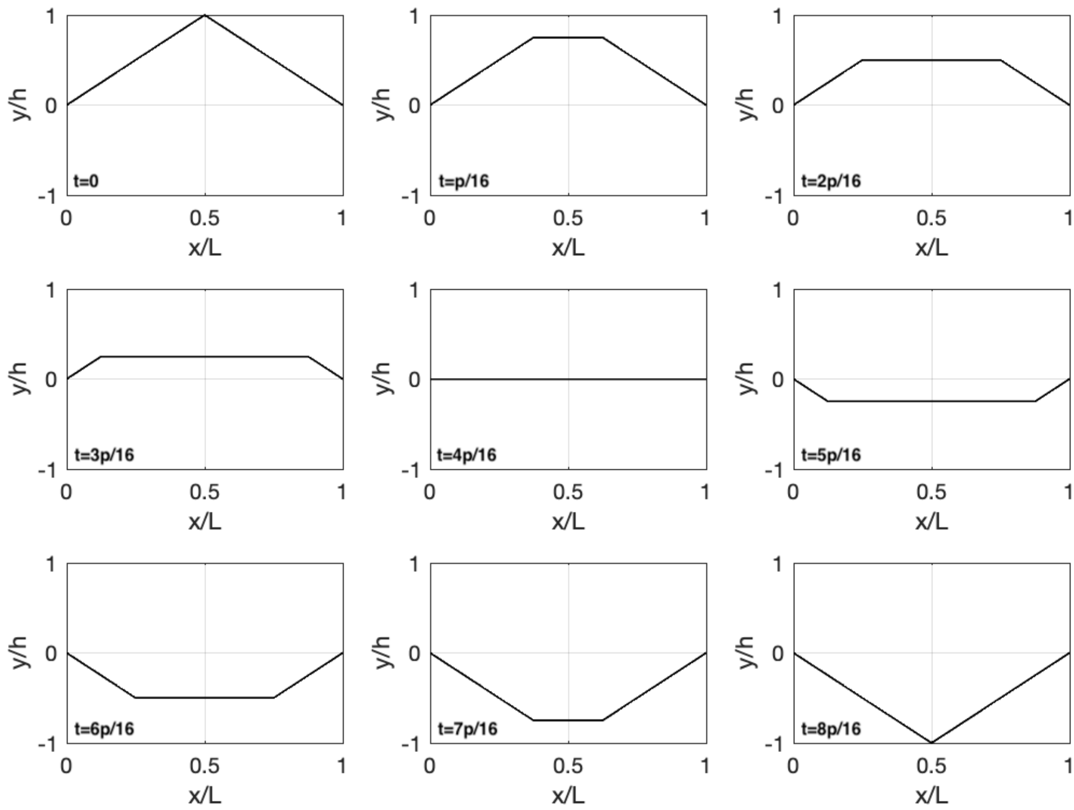
$$\frac{n\pi c}{L}b_n = \frac{2}{L}\int_{L_1}^{L_2} v\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{2v}{n\pi}\left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right],$$

giving

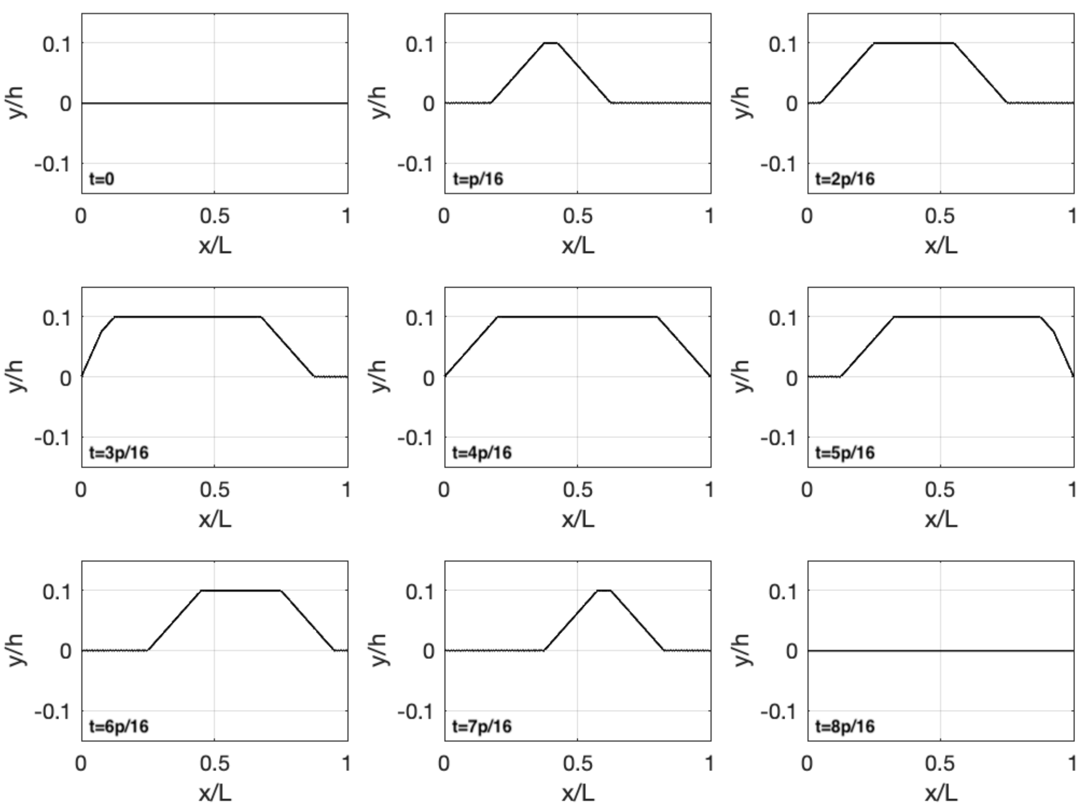
$$y(x,t) = \frac{2vL}{c\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

- We plot below snapshots of the evolution of the guitar and piano string (the latter for $L_1/L = 0.3$, $L_2/L = 0.5$) over the first half-period of the oscillation (with p being the prime period of the oscillation), which illustrates the persistence of corners moving with speed c.

Example: guitar string



Example: piano string



Conservation of energy

- Suppose an elastic string is stretched between $x = 0$ and $x = L$ along the x-axis to a line density ρ and a tension T, so that its small transverse displacement $y(x,t)$ is governed by the wave equation (1) and the boundary conditions (2), with wave speed $c = \sqrt{T/\rho}$.

- Recall that the point of the string that lies at $x\mathbf{i}$ in its so-called reference configuration is displaced transversely to the point with position vector $\mathbf{r}(x,t) = x\mathbf{i}+y(x,t)\mathbf{j}$. We note that when we impose the initial conditions (3), we must deform the string from its reference configuration along the x-axis to have transverse displacement $y(x,0) = f(x)$ and we must impart on the string the transverse velocity given by $y_t(x,0) = g(x)$.

- The kinetic energy of the string is therefore given by

$$\int_0^L \frac{1}{2}\rho\,|\boldsymbol{r}_t|^2\,\mathrm{d}x = \int_0^L \frac{1}{2}\rho y_t^2\,\mathrm{d}x.$$

- Moreover, the elastic potential energy of the string is the product of tension and extension, and therefore given by

$$T\left(\int_0^L |\boldsymbol{r}_x|\,\mathrm{d}x - L\right) = T\int_0^L \left(1+y_x^2\right)^{\frac{1}{2}} - 1\,\mathrm{d}x.$$

Since the transverse displacement is small in the sense that $|y_x| \ll 1$, a Taylor expansion gives

$$\left(1+y_x^2\right)^{\frac{1}{2}} - 1 = \frac{1}{2}y_x^2 + \cdots\,,$$

so to a first approximation (*i.e.* neglecting cubic and higher order terms), the elastic potential energy is given by

$$\int_0^L \frac{1}{2}Ty_x^2\,\mathrm{d}x.$$

- The energy of a string is defined to be the sum of its kinetic and elastic potential energies, and hence given by

$$E(t) = \int_0^L \frac{1}{2}\rho y_t^2 + \frac{1}{2}Ty_x^2\,\mathrm{d}x.$$

- If $y(x,t)$ satisfies the wave equation (1) and the boundary conditions (2), then $E(t)$ is constant for $t > 0$ because

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_0^L \rho y_t y_{tt} + T y_x y_{xt}\,\mathrm{d}x && \text{(by LIR)}\\
&= \int_0^L T y_t y_{xx} + T y_x y_{xt}\,\mathrm{d}x && \text{(by (1) \& } c^2 = T/\rho)\\
&= \int_0^L (T y_t y_x)_x\,\mathrm{d}x\\
&= [T y_t y_x]_{x=0}^{x=L}\\
&= 0\,,
\end{aligned}$$

where in final equality we used the fact that each of the boundary conditions may be differentiated with respect to t to deduce that $y_t(0,t) = y_t(L,t) = 0$ for $t > 0$.

Remarks

(1) We have shown that the energy of the elastic string is conserved during its motion, with the kinetic and elastic potential energy being transferred back and forth as the string oscillates.

(2) The energy of the string is set by the initial conditions in (3) to be given by

$$E(t) = E(0) = \int_0^L \frac{1}{2}\rho(g(x))^2 + \frac{1}{2}T(f'(x))^2 \, \mathrm{d}x.$$

(3) The energy of the nth normal mode $y_n(x,t)$ is given by

$$E_n(t) = \int_0^L \frac{1}{2}\rho\left(\frac{\partial y_n}{\partial t}\right)^2 + \frac{1}{2}T\left(\frac{\partial y_n}{\partial x}\right)^2 \mathrm{d}x.$$

Since $y_n(x,t)$ satisfies (1) and (2) by construction, it follows that its energy is conserved during its motion and given by

$$E_n(t) = E_n(0) = \frac{n^2\pi^2\rho c^2 b_n^2}{4L} + \frac{n^2\pi^2 T a_n^2}{4L},$$

where in the last equality we substituted for $y_n(x,0)$ and integrated.

(4) Assuming convergence, Parseval's Identity for g and f' imply that

$$\int_0^L \frac{1}{2}\rho g(x)^2 + \frac{1}{2}Tf'(x)^2 \, \mathrm{d}x = \sum_{n=1}^{\infty}\left(\frac{n^2\pi^2\rho c^2 b_n^2}{4L} + \frac{n^2\pi^2 T a_n^2}{4L}\right),$$

and hence that

$$E(t) = E(0) = \sum_{n=1}^{\infty} E_n(0) = \sum_{n=1}^{\infty} E_n(t),$$

i.e. the energy of the elastic string is made up of that in its normal modes.

Uniqueness

- **Uniqueness Theorem:** The initial boundary value problem (1)–(3) has only one solution.
- **Proof of the uniqueness theorem:** Our strategy is to show that the difference between any two solutions much vanish.
- We suppose that $y(x,t)$ and $\widetilde{y}(x,t)$ are solutions to (1)–(3) and let

$$w(x,t) = y(x,t) - \widetilde{y}(x,t)$$

be their difference.

- By linearity, (1)-(3) imply that $w(x,t)$ satisfies the wave equation

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0, \tag{1$'$}$$

with the boundary conditions

$$w(0,t) = 0, \quad w(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2$'$}$$

and the initial conditions

$$w(x,0) = 0, \quad \frac{\partial w}{\partial t}(x,0) = 0 \quad \text{for} \quad 0 < x < L, \tag{3$'$}$$

- Since w is the small transverse displacement of an elastic string whose initial transverse displacement and velocity are everywhere zero and whose ends are fixed thereafter, on physical grounds we expect the string to remain stationary along the x-axis, *i.e.* $w = 0$ for $0 \leq x \leq L$ and $t \geq 0$, which is what we need to show to prove uniqueness.

- The trick now is to analyse the energy $E(t)$ associated with $w(x,t)$, which is given by

$$E(t) = \int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2} T w_x^2 \,\mathrm{d}x.$$

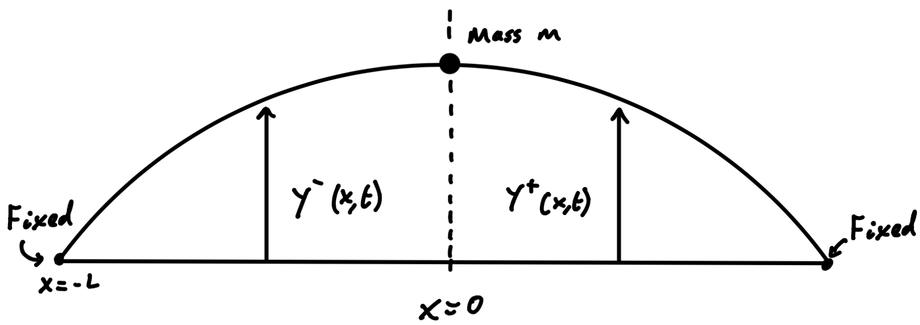
- Since w satisfies (1$'$) and (2$'$), the energy $E(t)$ is conserved. But $E(0) = 0$ by (3$'$), so

$$\int_0^L \frac{1}{2}\rho w_t^2 + \frac{1}{2} T w_x^2 \,\mathrm{d}x = 0 \quad \text{for} \quad t \geq 0.$$

- We deduce that $w_t = w_x = 0$ for $0 < x < L$, $t > 0$ assuming W_t and W_x are continuous on the region $R = \{(x,y) : 0 \leq x \leq L,\ t \geq 0\}$. Since (2$'$) and (3$'$) imply that $w = 0$ on the boundary of R, we deduce that $w = 0$ or $y = \widetilde{y}$ on R, which completes the proof.

Normal modes for a weighted string

- An elastic string of length $2L$ has its ends fixed at $(x,y) = (\pm L, 0)$ and a point particle of mass m is attached to the mid-point, as illustrated in the schematic below.



- Question: What are the normal modes of vibration?

- Since the tension T in the elastic string is assumed to be constant and the transverse displacements small (in the sense that $|y_x| \ll 1$), the horizontal forces exerted on the point particle by the string will balance to a first approximation, so we need only consider the transverse displacement of the point particle, $Y(t)$ say.

- We let $y^-(x,t)$ and $y^+(x,t)$ denote the small transverse displacements for $-L \leq x < 0$ and $0 < x \leq L$, respectively.

- Then y^- and y^+ must satisfy the wave equations

$$\frac{\partial^2 y^-}{\partial t^2} = c^2 \frac{\partial^2 y^-}{\partial x^2} \quad \text{for} \quad -L < x < 0, \tag{1^-}$$

$$\frac{\partial^2 y^+}{\partial t^2} = c^2 \frac{\partial^2 y^+}{\partial x^2} \quad \text{for} \quad 0 < x < L, \tag{1^+}$$

and the boundary conditions

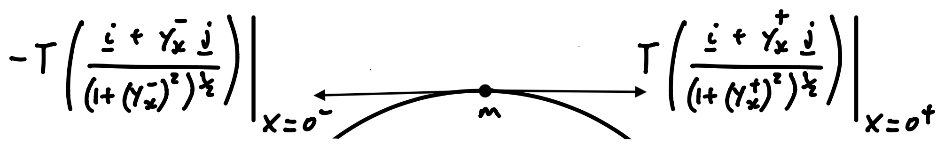
$$y^-(-L,t) = 0, \tag{2^-}$$

$$y^+(L,t) = 0. \tag{2^+}$$

- What conditions hold at $x = 0$? There are two. Firstly, since the point particle is attached to the string, we require

$$y^-(0_-,t) = Y(t) = y^+(0_+,t). \tag{3}$$

Secondly, the string exerts on the point particle the forces illustrated below (neglecting gravity and air resistance as in the wave equations above).



Hence, applying Newton's Second Law to the point particle in the y-direction gives

$$m\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} = \big(T\boldsymbol{\tau}(0_+,t) - T\boldsymbol{\tau}(0_-,t)\big)\cdot \mathbf{j},$$

where $\boldsymbol{\tau}$ is the right-pointing unit tangent vector that we recall to be given by

$$\boldsymbol{\tau} = \frac{\boldsymbol{i} + y_x\boldsymbol{j}}{(1+y_x^2)^{1/2}};$$

we deduce that

$$m\frac{\mathrm{d}^2 Y}{\mathrm{d}t^2} = Ty_x^+(0_+,t) - Ty_x^-(0_-,t), \tag{4}$$

to a first approximation for small transverse displacements for which $|y_x| \ll 1$.

- To find the normal modes we seek nontrivial separable solutions of (1)–(4) of the form

$$y^{\pm} = F_{\pm}(x)G(t),$$

since we must choose the same time dependence for both y^- and y^+ in order to satisfy (3).

- In the usual manner we may deduce from ($1^{\pm}$) that there is a real constant λ such that

$$\frac{F_{\pm}''(x)}{F_{\pm}(x)} = \frac{G''(t)}{c^2 G(t)} = -\lambda. \tag{$I^{\pm}$}$$

- Since we're seeking nontrivial solutions, it follows from ($2^{\pm}$) that

$$F_-(-L) = 0, \;\; F_+(L) = 0. \tag{$II^{\pm}$}$$

- Similarly (3) gives

$$F_-(0_-) = F_+(0_+), \qquad (III)$$

while (4) implies

$$mF_\pm(0)G''(t) = T(F'_+(0_+) - F'_-(0_-))G(t);$$

eliminating the time dependence in this expression using $(I^\pm)$ and eliminating T via $T = \rho c^2$ gives

$$-\lambda\frac{m}{\rho}F_\pm(0) = F'_+(0_+) - F'_-(0_-). \qquad (IV)$$

- It may be shown that there are no nontrivial solutions of $(I^\pm)$–(IV) for $\lambda \leq 0$, and so we seek nontrivial solutions for $\lambda = \omega^2$, where $\omega > 0$ without loss of generality.

- Since then $G'' + \omega^2 c^2 G = 0$, $(I^\pm)$ gives $G(t) = C\cos(\omega ct + \epsilon)$, where we may take $C = 1$ without loss of generality and ϵ is an arbitrary constant, *i.e.* oscillatory solutions with frequency ωc.

- Moreover, $(I^\pm)$ give

$$\begin{aligned} F''_- + \omega^2 F_- &= 0 \text{ for } -L < x < 0, \\ F''_+ + \omega^2 F_+ &= 0 \text{ for } 0 < x < L, \end{aligned}$$

so that $(II^\pm)$ imply

$$\begin{aligned} F_-(x) &= A\sin\big(\omega(L+x)\big), \\ F_+(x) &= B\sin\big(\omega(L-x)\big), \end{aligned}$$

where A and B are arbitrary real constants.

- Substituting these expressions for $F_\pm(x)$ into (III) and (IV), we obtain two linear algebraic equations for A and B that may be written in the form

$$\underbrace{\begin{bmatrix} \sin\omega L & -\sin\omega L \\ \cos\omega L - \frac{m\omega}{\rho}\sin\omega L & \cos\omega L \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \qquad (\dagger)$$

- For nontrivial solutions for $F_\pm(x)$, we need

$$\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and hence for the matrix M to be singular: setting $\det M = 0$, we deduce that ω must satisfy

$$\sin\omega L\left(2\cos\omega L - \frac{m\omega}{\rho}\sin\omega L\right) = 0.$$

- Hence, there are two cases:

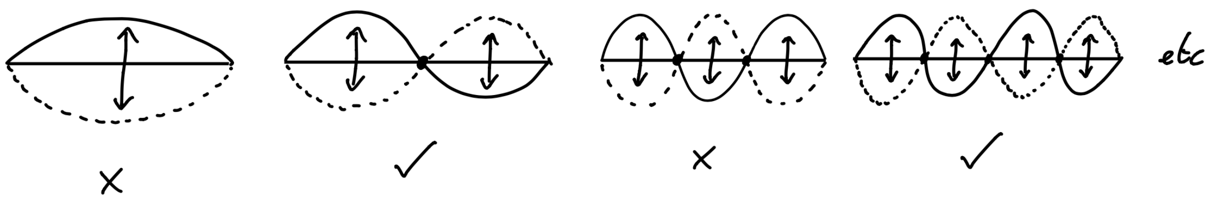
  (i) $\sin\omega L = 0$;

  (ii) $2\cos\omega L - \dfrac{m\omega}{\rho}\sin\omega L = 0$.

- In case (i) we deduce that $\omega = n\pi/L$, where n is a positive integer, so that (†) implies $B = -A$ and hence the normal modes

$$\begin{aligned} y_-(x,t) &= \quad A\sin\big(\omega(L+x)\big)\cos(\omega ct+\epsilon), \\ y_+(x,t) &= -A\sin\big(\omega(L-x)\big)\cos(\omega ct+\epsilon). \end{aligned}$$

This means that the normal modes are the same as for a string of length $2L$ with a node at $x = 0$, *i.e.* the point particle is stationary and remains at the origin, as illustrated for the first few such models in the schematic below.



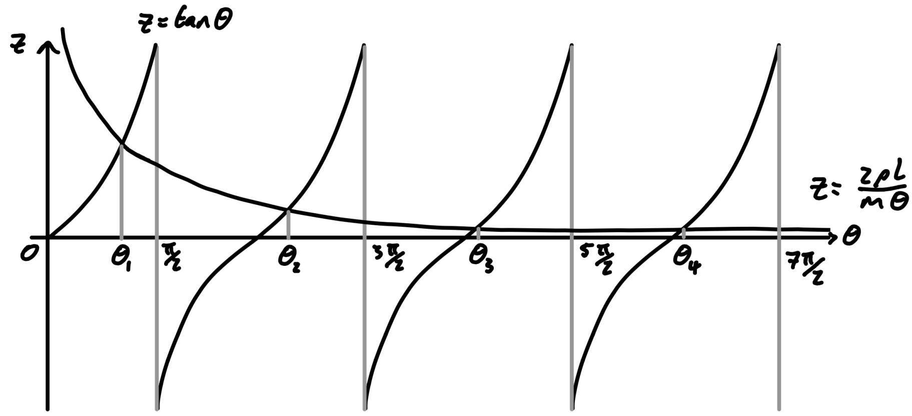
- In case (ii), we let $\omega = \theta/L$, so that θ satisfies the equation

$$\tan\theta = \frac{\alpha}{\theta}, \tag{‡}$$

where the dimensionless parameter $\alpha = 2L\rho/m$ is the ratio of the mass of the string to that of the point particle. By plotting the graph of the left- and right-hand sides of (‡), as illustrated below, we can convince ourselves that there are countably many roots

$$\theta_1 < \theta_2 < \theta_3 < \cdots$$

for θ, and hence countably many natural frequencies $\omega c = \theta_n c/L$, where n is a positive integer.



Now (†) implies that $B = A$ and hence the normal modes

$$\begin{aligned} y_-(x,t) &= A\sin\big(\omega(L+x)\big)\cos(\omega ct+\epsilon), \\ y_+(x,t) &= A\sin\big(\omega(L-x)\big)\cos(\omega ct+\epsilon), \end{aligned}$$

which means that the normal modes are symmetric about $x = 0$, as illustrated for the first few such modes in the schematic below.