# Fourier Series & PDEs: Lectures 11-12

## General solution to the wave equation

- It is a remarkable fact that it is possible to write down all solutions of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

where we recall that the parameter $c > 0$ is the wave speed.

- To verify this fact we introduce new independent variables

$$\xi = x - ct, \quad \eta = x + ct,$$

and seek a solution in which

$$y(x,t) = Y(\xi, \eta).$$

- The chain rule implies

$$y_x = Y_\xi \xi_x + Y_\eta \eta_x = Y_\xi + Y_\eta,$$

$$y_t = Y_\xi \xi_t + Y_\eta \eta_t = -cY_\xi + cY_\eta,$$

and

$$y_{xx} = (Y_\xi + Y_\eta)_\xi \xi_x + (Y_\xi + Y_\eta)_\eta \eta_x = Y_{\xi\xi} + 2Y_{\xi\eta} + Y_{\eta\eta},$$

$$y_{tt} = (-cY_\xi + cY_\eta)_\xi \xi_t + (-cY_\xi + cY_\eta)_\eta \eta_t = c^2(Y_{\xi\xi} - 2Y_{\xi\eta} + Y_{\eta\eta}),$$

where we assumed $Y_{\xi\eta} = Y_{\eta\xi}$. Hence,

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = -4c^2 \frac{\partial^2 Y}{\partial \xi \partial \eta}.$$

- Hence in the new variables the wave equation transforms to the equation

$$\frac{\partial^2 Y}{\partial \xi \partial \eta} = 0,$$

*i.e.*

$$\frac{\partial}{\partial \xi}\left(\frac{\partial Y}{\partial \eta}\right) = 0.$$

Thus $\partial Y/\partial \eta$ is independent of $\xi$ and is a function of $\eta$ only, say $G'(\eta)$, *i.e.*

$$\frac{\partial Y}{\partial \eta} = G'(\eta),$$

and so

$$\frac{\partial}{\partial \eta}\left[Y - G(\eta)\right] = 0.$$

Thus, $Y - G(\eta)$ is a function of $\xi$ only, say $F(\xi)$, and therefore

$$Y - G(\eta) = F(\xi),$$

giving

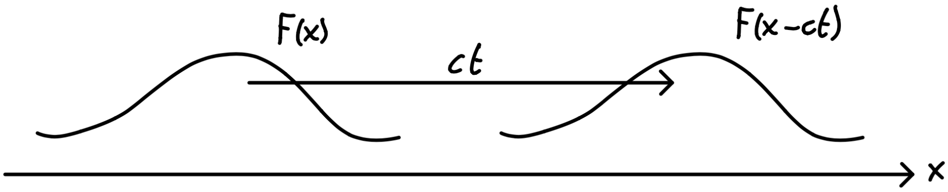
$$y(x,t) = F(x - ct) + G(x + ct), \qquad (\star)$$

where $F$ and $G$ are arbitrary twice continuously differentiable functions.

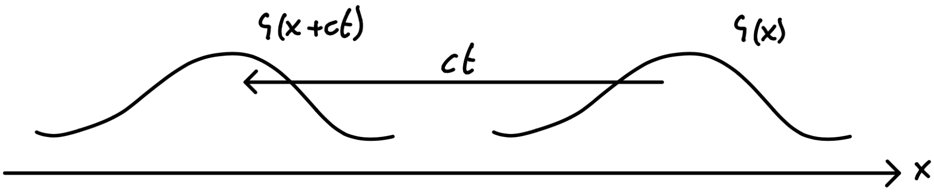
**Remarks**

(1) It is easy to verify that $(\star)$ is a solution. We have shown that all solutions must be of this form.

(2) We note that $F(x-ct)$ is a travelling wave of constant shape moving in the positive $x$-direction with speed $c$, as illustrated in the sketch below in which the initial profile $y = F(x)$ at $t = 0$ is translated a distance $ct$ to the right at time $t$.




(3) We note that $G(x+ct)$ is a travelling wave of constant shape moving in the negative $x$-direction with speed $c$, as illustrated in the sketch below in which the initial profile $y = G(x)$ at $t = 0$ is translated a distance $ct$ to the left at time $t$.




(4) The general solution is therefore the superposition of left- and right-travelling waves each moving with speed $c$, which is the reason the parameter $c$ is called the wave speed. It follows that the wave equation propagates information at constant speed $c$ in contrast to solutions of the heat equation in which information propagates at infinite speed. Examples of the latter are the fundamental solution of the heat equation in question 2 of sheet 3 (which is positive and has constant thermal energy for $t > 0$, but with the property that $T(x,t) \to 0$ as $t \to 0+$ for $x \neq 0$, so that the influence of the "point source of heat" concentrated at the origin at $t = 0$ is felt everywhere for $t > 0$) and the inhomogeneous Dirichlet problem in lecture 7 (in which the temperature is positive everywhere for $t > 0$, while being equal to zero at $t = 0$ for $0 < x < L$).

**Waves on an infinite string: D'Alembert's formula**

- Consider the initial boundary value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty,\ t > 0, \tag{1}$$

with the initial conditions

$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty, \tag{2}$$

where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.

- The general solution of the heat equation (1) is given by $(\star)$, so it remains to determine the functions $F$ and $G$ for which it satisfies the initial conditions (2).

- Substituting $(\star)$ into (2) gives

$$F(x) + G(x) = f(x), \tag{a}$$

$$-cF'(x) + cG'(x) = g(x). \tag{b}$$

The expressions ($b$) integrates to give

$$-F(x)+G(x)=\frac{1}{c}\int_0^x g(s)\,\mathrm{d}s+A, \tag{c}$$

where $A$ is an arbitrary constant. Subtracting and adding ($a$) and ($c$), we deduce that $F$ and $G$ are given by

$$F(x)=\frac{1}{2}\left[f(x)-\frac{1}{c}\int_0^x g(s)\,\mathrm{d}s-a\right],$$
$$G(x)=\frac{1}{2}\left[f(x)+\frac{1}{c}\int_0^x g(s)\,\mathrm{d}s+a\right].$$

Thus,

$$\begin{aligned} y(x,t) &= \frac{1}{2}\left(f(x-ct)-\frac{1}{c}\int_0^{x-ct} g(s)\,\mathrm{d}s-a\right)+\frac{1}{2}\left(f(x+ct)+\frac{1}{c}\int_0^{x+ct} g(s)\,\mathrm{d}s+a\right) \\ &= \frac{1}{2}\big(f(x-ct)+f(x+ct)\big)+\frac{1}{2c}\left(\int_{x-ct}^{0} g(s)\,\mathrm{d}s+\int_0^{x+ct} g(s)\,\mathrm{d}s\right) \end{aligned}$$

and we arrive at D'Alembert's Formula

$$y(x,t)=\frac{1}{2}\big(f(x-ct)+f(x+ct)\big)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s. \tag{DF}$$

**Remarks**

(1) The argument shows that, for given $f$ and $g$, the initial value problem has one and only one solution, *i.e.* existence and uniqueness.

(2) We note that uniqueness may also be proved by energy conservation under the additional assumption that $y_t$, $y_x \to 0$ sufficiently rapidly as $x \to \pm\infty$ that we can ensure the existence of the energy

$$E(t)=\int_{-\infty}^{\infty} \frac{\rho}{2}y_t^2+\frac{T}{2}y_x^2\ \mathrm{d}x.$$

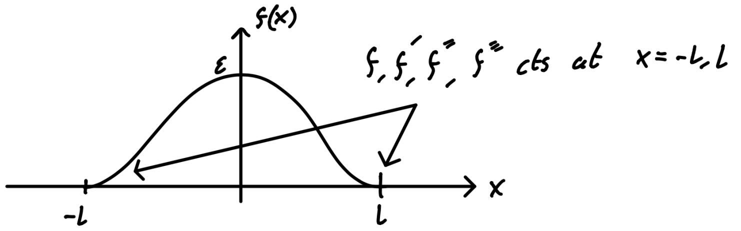
**Example 1**

- Suppose that $f$ and $g$ are given by

$$f(x)=\begin{cases} \epsilon\cos^4\left(\dfrac{\pi x}{2L}\right) & \text{for } |x|\le L, \\ 0 & \text{otherwise,} \end{cases} \qquad g(x)=0,$$

  where $\epsilon$ and $L$ are positive constants.

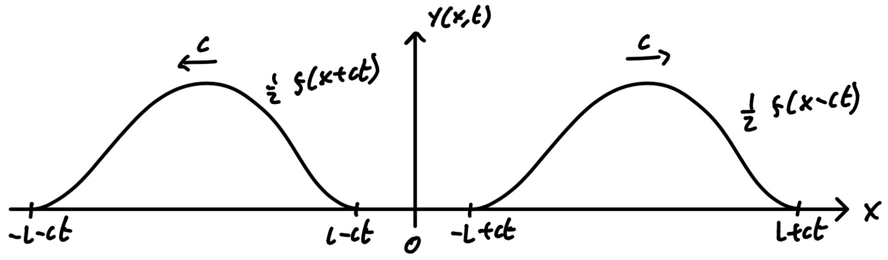
- We note that $f$ is said to be compactly supported because it is only non-zero on a closed bounded interval, namely $[-L,L]$, and that $f$ and its first three derivatives $f'$, $f''$ and $f'''$ are continuous on $R$, as illustrated in the sketch below of the graph of $f$.



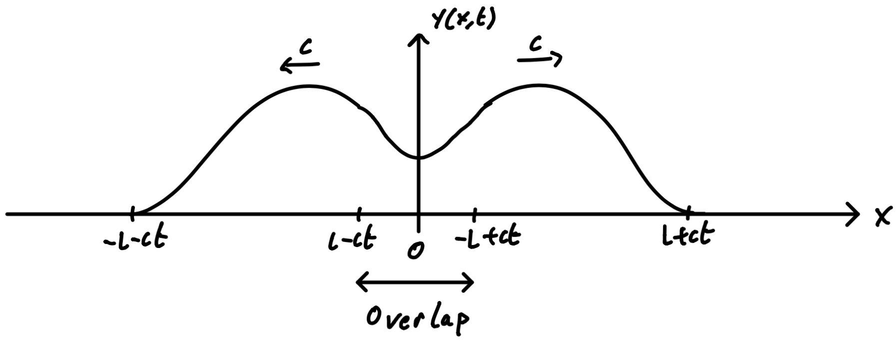
- By D'Alembert's formula (DF) the solution of the initial boundary value problem is given by

$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big),$$

- We note that this is a so-called classical solution because it is twice continuously differentiable with respect to $x$ and $t$ and satisfies the wave equation.
- We can sketch the solution $y(x,t)$ at a fixed time $t > 0$ using the geometrical properties of its travelling wave components.
  - For $ct > L$, the supports of $f(x-ct)$ and $f(x+ct)$ do not overlap, as illustrated below.




  - For $0 < ct < L$, the supports of $f(x-ct)$ and $f(x+ct)$ overlap, as illustrated below.




- The derivation of explicit formulae for the solution therefore requires some careful book keeping for which it is much easier to think geometrically rather than algebraically.

**Characteristic diagram**

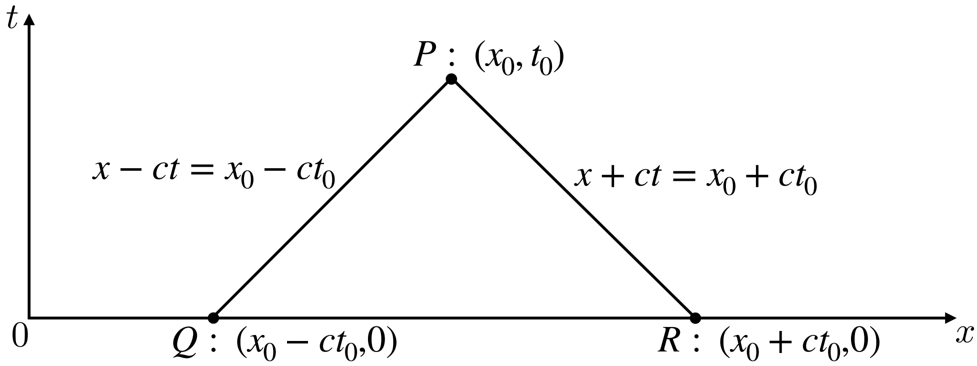
- Let us ask how the solution at a point $P:\ (x_0, t_0)$ in the upper half of the $(x,t)$-plane depends upon the data $f$ and $g$.
- By D'Alembert's Formula (DF), we have

$$y(x_0,t_0) = \frac{1}{2}[f(x_0 - ct_0) + f(x_0 + ct_0)] + \frac{1}{2c}\int_{x_0 - ct_0}^{x_0 + ct_0} g(x)\,\mathrm{d}x,$$

which may be written in the form

$$y(P) = \frac{1}{2}\big(f(Q) + f(R)\big) + \frac{1}{2c}\int_Q^R g(s)\,\mathrm{d}s, \qquad \text{(DFB)}$$

where $Q$ and $R$ are the points $(x_0 - ct_0, 0)$ and $(x_0 + ct_0, 0)$, respectively, on the $x$-axis, as illustrated in the sketch below.

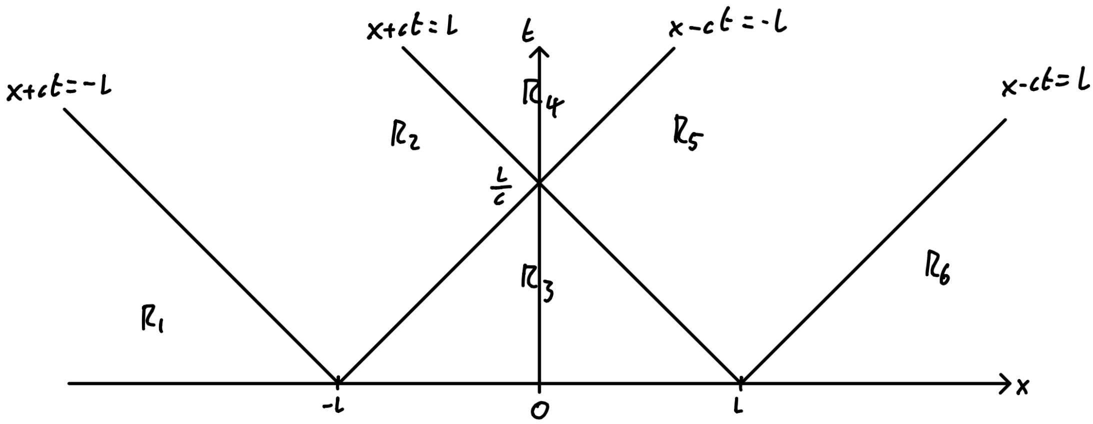



- We note the deliberate abuse of notation in (DFB) to aid the geometric interpretation of (DF).
- **Definition:** The lines $x \pm ct = x_0 \pm ct_0$ are the characteristic lines through $P:\ (x_0, t_0)$.
- It follows from (DFB) that $y(P)$ depends only on
  - (i) $f$ though the values $f$ takes at $Q$ and $R$;
  - (ii) $g$ though the values $g$ takes on the $x$-axis between $Q$ and $R$.

  This motivates the following definition.
- **Definition:** The interval $[x_0 - ct_0, x_0 + ct_0]$ of the $x$-axis between $Q$ and $R$ is called the domain of dependence of $P:\ (x_0, t_0)$
- If $f$ or $g$ are modified outside the domain of dependence of $P$, then $y(P)$ is unchanged.
- We can exploit the geometric interpretation of (DF') to construct explicit formulae for the solution: the contribution to $y(P)$ from $f$ and $g$ changes at points on the $x$-axis where $f$ and $g$ change their analytic behaviour.
- Hence, given a particular $f$ and $g$, the first task is to identify these points on the $x$-axis and sketch the characteristic lines $x \pm ct =$ constant through each of them — this is the characterisrtic diagram.
- The characteristic diagram divides the $(x, t)$-plane into regions in which the contributions from $f$ and $g$ may be different: the second task is to evaluate $y(P)$ for $P$ in each of these regions.

**Example 1 revisited**

- In this case $f(x)$ changes its analytic behaviour at the points $(-L, 0)$ and $(L, 0)$. We construct the characteristics through these points and thus divide up the upper-half of the $(x, t)$-plane into six regions $R_1, \ldots, R_6$, forming the characteristic diagram illustrated below.



- In particular, in $t > 0$, the region $R_1$ is below $x + ct = -L$; the region $R_2$ is above $x + ct = -L$ and above $x - ct = -L$; the region $R_3$ is below $x - ct = -L$ and below $x + ct = L$; the region $R_4$ is above $x + ct = L$ and above $x - ct = -L$; the region $R_5$ is above $x + ct = L$ and above $x - ct = L$; and the region $R_6$ is below $x - ct = L$).

- By (DFB) we have

$$y(P) = \frac{1}{2}\big(f(Q) + f(R)\big).$$

- Since $PQ$ is parallel to the characteristics $x - ct = \pm L$, while $PR$ is parallel to the characteristics $x + ct = \pm L$, the solution is as follows:

  - $P \in R_1 \implies \implies y(x,t) = \frac{1}{2}[0 + 0]$;
  - $P \in R_2 \implies y(x,t) = \frac{1}{2}\left[0 + \epsilon\cos^4\left(\frac{\pi}{2c}(x + ct)\right)\right]$;
  - $P \in R_3 \implies y(x,t) = \frac{1}{2}\left[\epsilon\cos^4\left(\frac{\pi}{2c}(x - ct)\right) + \epsilon\cos^4\left(\frac{\pi}{2c}(x + ct)\right)\right]$;
  - $P \in R_4 \implies y(x,t) = \frac{1}{2}[0 + 0]$;
  - $P \in R_5 \implies y(x,t) = \frac{1}{2}\left[\epsilon\cos^4\left(\frac{\pi}{2c}(x - ct)\right) + 0\right]$;
  - $P \in R_6 \implies y(x,t) = \frac{1}{2}[0 + 0]$.

  For example,

  - when $P \in R_1$, both $Q$ and $R$ lie to the left of $(-L, 0)$, so f(Q) = f(R) = 0;
  - when $P \in R_2$, $Q$ is to the left of $(-L, 0)$, while $R$ lies between $(-L, 0)$ and $(L, 0)$, so $f(Q) = 0$ and $f(R) = f(x + ct) = \epsilon\cos^4\left(\frac{\pi}{2c}(x + ct)\right)$;
  - when $P \in R_3$, both $Q$ and $R$ are between $(-L, 0)$ and $(L, 0)$, so $f(Q) = f(x - ct) = \epsilon\cos^4\left(\frac{\pi}{2c}(x - ct)\right)$ and $f(R) = f(x + ct) = \epsilon\cos^4\left(\frac{\pi}{2c}(x + ct)\right)$; *etc.*

- We note that since $y$ is continuous on characteristics bounding the regions, it does not matter to which region each belongs when it comes to writing out the solution everywhere in $t > 0$, *e.g.* we could pick $R_1: \; x + ct < -L,\ t > 0$; $R_2: \; -L \leq x + ct \leq L,\ x - ct \leq L$; $R_3: \; -L < x + ct < L$, $-L < x - ct < L,\ t > 0$; *etc.*

**Example 2**

- Suppose that $f$ and $g$ are given by

$$f(x) = 0, \qquad g(x) = \begin{cases} vx/L & \text{for } |x| \leq L\,, \\ 0 & \text{otherwise,} \end{cases}$$

  where $L$ and $v$ are positive constants.

- By (DFB) we have

$$y(P) = \frac{1}{2c}\int_Q^R g(s)\,\mathrm{d}s.$$

- Since $PQ$ is parallel to the characteristics $x - ct = \pm L$, while $PR$ is parallel to the characteristics $x + ct = \pm L$, the solution is as follows:

$$R_1: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0$$

$$R_2: \quad y = \frac{1}{2c}\int_{x-ct}^{-L} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-L}^{x+ct} \frac{vs}{L}\,\mathrm{d}s = \frac{v}{4Lc}\left((x+ct)^2 - L^2\right)$$

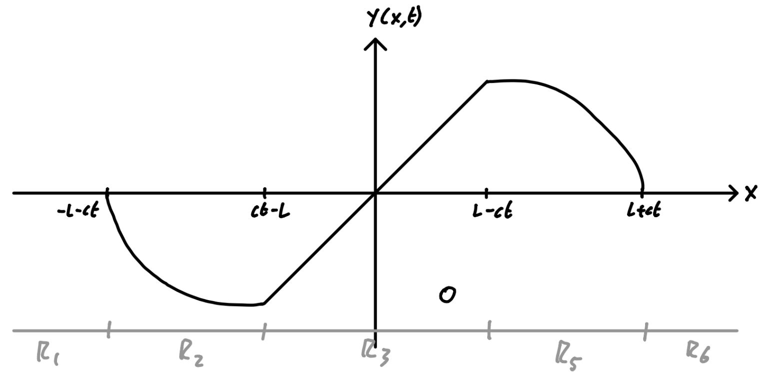
$$R_3: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{vs}{L}\,\mathrm{d}s = \frac{v}{4Lc}\left((x+ct)^2 - (x-ct)^2\right) = \frac{vxt}{L}$$

$$R_4: \quad y = \frac{1}{2c}\int_{x-ct}^{-L} 0\,\mathrm{d}s + \frac{1}{2c}\int_{-L}^{L} \frac{vs}{L}\,\mathrm{d}s + \frac{1}{2c}\int_{L}^{x+ct} 0\,\mathrm{d}s = 0$$

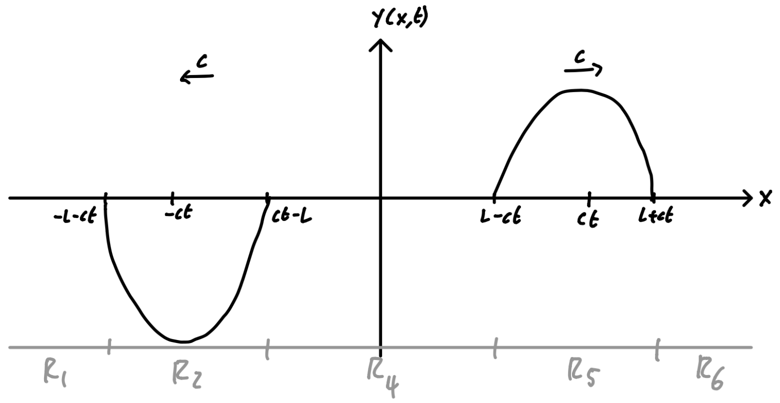
$$R_5: \quad y = \frac{1}{2c}\int_{x-ct}^{L} \frac{vs}{L}\,\mathrm{d}s + \frac{1}{2c}\int_{L}^{x+ct} 0\,\mathrm{d}s = \frac{v}{4Lc}\left(L^2 - (x-ct)^2\right)$$

$$R_6: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,\mathrm{d}s = 0$$

- We can use the solution to plot $y$ as a function of $x$ at different times. For example, for $0 < t < L/c$, the profile is as follows.




- However, for $t > L/c$, the profile is as follows:



- As $t$ increases we see two packets of displacement, one moving to the left with speed $c$ and the other to the right with speed $c$. Away from them the displacement is zero.
- We note that the solution we have constructed is not a classical one because the profile has corners on the characteristics through $(\pm L, 0)$. The solution is however infinitely differentiable away from these corners (the profile being made up of segments of straight lines and parabolae).