Fourier Series & PDEs: Lectures 13-14

Laplace's equation

Derivation of the three-dimensional heat equation

- We begin by recalling from Multivariable Calculus the derivation of the three-dimensional heat equation because it introduces all of the quantities that we shall need.

- Let $T(\mathbf{x}, t)$ be the absolute temperature in a rigid isotropic conducting material (*e.g.* metal), with constant density ρ and specific heat c_v.

- Let $\mathbf{q}(\mathbf{x}, t)$ be the heat flux vector, so that $\mathbf{q} \cdot \mathbf{n}\, \mathrm{d}S$ is the rate at which thermal energy is transported through a surface element $\mathrm{d}S$ in the direction of a unit normal $\mathbf{n}$ to $\mathrm{d}S$.

- Let V be a fixed region in the medium with boundary ∂V and let $\mathbf{n}$ be the outward unit normal to ∂V. Assuming there are no sources or sinks of thermal energy, conservation of thermal energy in V is given by
$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho c_v T \, \mathrm{d}V = \iint_{\partial V} \mathbf{q} \cdot (-\mathbf{n}) \, \mathrm{d}S,$$
where the term on the left-hand side is the rate of change of thermal energy in V, while the term on the right-hand side is the net rate at which thermal energy enters V through ∂V.

- Differentiating under the integral sign on the left-hand side and applying the Divergence Theorem on the right-hand side gives
$$\iiint_V \left(\rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{q} \right) \mathrm{d}V = 0.$$

- Since V is arbitrary, the integrand must be zero (if it is continuous), *i.e.*
$$\rho c_v \frac{\partial T}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{q} = 0.$$

- A closed model for heat conduction is obtained by prescribing a constitutive law relating the heat flux vector $\mathbf{q}$ to the temperature T. Fourier's Law states that thermal energy is transported down the temperature gradient, with $\mathbf{q} = -k\boldsymbol{\nabla}T$, where k is the constant thermal conductivity.

- Hence, T satisfies the three-dimensional heat or diffusion equation given by
$$\frac{\partial T}{\partial t} = \kappa \boldsymbol{\nabla}^2 T,$$
where the thermal diffusivity $\kappa = k/\rho c_v$.

- The SI units of the dependent variables and dimensional parameters are summarized in the following table.

Quantity	Symbol	SI units
Temperature	T	K
Heat flux vector	$\mathbf{q}$	$\mathrm{J\,m^{-2}\,s^{-1}}$
Density	ρ	$\mathrm{Kg\,m^{-3}}$
Specific heat	c_v	$\mathrm{J\,Kg^{-1}\,K^{-1}}$
Thermal conductivity	k	$\mathrm{J\,m^{-1}\,s^{-1}\,K^{-1}}$
Thermal diffusivity	κ	$\mathrm{m^2\,s^{-1}}$

Steady two-dimensional heat conduction

- In this course we consider two-dimensional steady-state solutions of the heat equation.
- Setting $T = T(x, y)$, the three-dimensional heat equation becomes

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

  i.e. $T(x, y)$ is governed by Laplace's equation in the plane.

- If r and θ are the usual plane polar coordinates, with $(x, y) = (r\cos\theta, r\sin\theta)$, Laplace's equation in the plane becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{for} \quad r > 0.$$

- We will use Fourier's method to construct solutions to several boundary value problems for Laplace's equation in the plane.

Boundary value problem in Cartesian coordinates

- An infinite straight metal rod has a rectangular cross-section whose sides are of length a and b. The temperature $T(x, y)$ in each cross-section satisfies the boundary value problem given by Laplace's equation

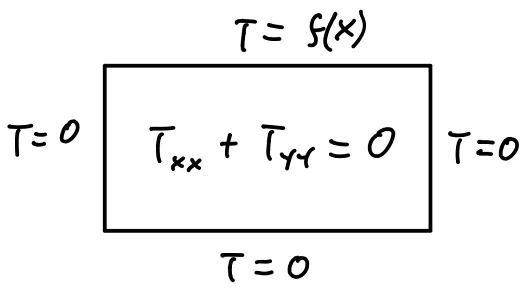
$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < L,\ 0 < y < L, \tag{1}$$

  with the boundary conditions

$$\begin{aligned} T(0, y) &= 0 \quad \text{for} \quad 0 < y < b, & (2)\\ T(a, y) &= 0 \quad \text{for} \quad 0 < y < b, & (3)\\ T(x, 0) &= 0 \quad \text{for} \quad 0 < x < a, & (4)\\ T(x, b) &= f(x) \quad \text{for} \quad 0 < x < a, & (5) \end{aligned}$$

  where $f(x)$ is the prescribed temperature at which the top face of the rod is held.

- We summarize the boundary value problem in the following figure.



- We note that in accordance with physical intuition, the temperature is prescribed on the boundary of the rectangle.
- We construct a solution to the boundary value problem using Fourier's method.

Step I

- We begin by finding all nontrivial separable solutions of Laplace's equation (1) and the boundary conditions (2)-(3) on the left- and right-hand sides of the rectangle.
- Substituting $T(x,y) = F(x)G(y)$ into (1) and dividing through by $F(x)G(y) \neq 0$ gives
$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}.$$
- The left-hand side of this expression is independent of y, while the right-hand side is independent of x. Since the left-hand side is equal to the right-hand side, they must both be independent of x and y, *i.e.* LHS = RHS = $-\lambda$ for some constant $\lambda \in \mathbb{R}$.
- Hence, $-F'' = \lambda F$ for $0 < x < a$, with (2) and (3) giving the boundary conditions $F(0) = 0$ and $F(a) = 0$ for nontrivial G.
- We solved this problem for F in Lecture 6: the nontrivial solutions are given for positive integers n by
$$F(x) = B\sin\left(\frac{n\pi x}{L}\right) \quad \text{with} \quad \lambda = \left(\frac{n\pi}{L}\right)^2,$$
where B is an arbitrary constant; since $G'' - \lambda G = 0$, the corresponding solution for $G(y)$ is given by
$$G = C\cosh\left(\frac{n\pi y}{a}\right) + D\sinh\left(\frac{n\pi y}{a}\right),$$
where C and D are arbitrary constants.
- Hence, the nontrivial separable solutions of (1) subject to (2)–(3) are given for positive integers n by
$$T_n(x,y) = \sin\left(\frac{n\pi x}{a}\right)\left(a_n\cosh\left(\frac{n\pi y}{a}\right) + b_n\sinh\left(\frac{n\pi y}{a}\right)\right),$$
where $a_n = BC$ and $b_n = BD$ are real constants.
- We note that in contrast to the wave equation for which the nontrivial separable solutions are the product of trigonometric functions in x and trigonometric functions in t, the nontrivial separable solutions of Laplace's equation are products of trigonometric functions in x with hyperbolic functions in y.

Step II

- Since (1)–(3) are linear, we can superimpose the separable solutions (assuming convergence) to obtain the general series solution
$$T(x,y) = \sum_{n=1}^{\infty} T_n(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\left(a_n\cosh\left(\frac{n\pi y}{a}\right) + b_n\sinh\left(\frac{n\pi y}{a}\right)\right).$$

Step III

- The boundary condition (4) on the bottom side of the rectangle can only be satisfied if $a_n = 0$ for all n, while the boundary condition (5) on the top side can only be satisfied if
$$f(x) = \sum_{n=1}^{\infty} b_n\sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right) \quad \text{for} \quad 0 < x < a,$$
so that the theory of Fourier series gives
$$b_n\sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x$$
for positive integers n.

Remarks

(1) We could have applied the boundary condition (4) on $y = 0$ at end of Step I.

(2) The case in which $a = b = L$ and $f = T^\star$ is a constant is considered on problem sheet 7.

Boundary value problem in plane polar coordinates

- Recall that in plane polar coordinates (r, θ), Laplace's equation for $T(r, \theta)$ becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \text{ for } r > 0 \qquad (\star)$$

- We start by finding all nontrivial separable solutions of the form $T(r, \theta) = F(r)G(\theta)$. Since T is a single-valued function of position on $r > 0$, we require $G(\theta)$ to be 2π-periodic.

- Substituting $T(r, \theta) = F(r)G(\theta)$ into $(\star)$ we obtain

$$F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0;$$

  dividing through by $F(r)G(\theta)/r^2 \neq 0$ gives

$$\frac{r^2 F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)}.$$

- The left-hand side of this expression is independent of θ, while the right-hand side is independent of r. Since the left-hand side is equal to the right-hand side, they must both be independent of r and θ, *i.e.* LHS = RHS = λ for some constant $\lambda \in \mathbb{R}$.

- Hence, we need to find all $\lambda \in \mathbb{R}$ for which $G''(\theta) + \lambda G(\theta) = 0$ has a nontrivial, 2π-periodic, solution $G(\theta)$. We consider cases.

<u>Case (i) $\lambda = -\omega^2$ ($\omega > 0$ wlog)</u>

- If $G'' - \omega^2 G = 0$, then $G(\theta) = A\cosh\omega\theta + B\sinh\omega\theta$, where A, $B \in \mathbb{R}$.

- If G is 2π periodic, then $G(0) = G(\pm 2\pi)$, which implies $A = A\cosh 2\pi\omega \pm B\sinh 2\pi\omega$, so that $A(\cosh 2\pi\omega - 1) = 0$ and $B\sinh 2\pi\omega = 0$, giving $A = B = 0$ and $G = 0$.

<u>$\lambda = 0$</u>

- If $G'' = 0$, then $G(\theta) = A + B\theta$, where A, $B \in \mathbb{R}$.

- If G is 2π periodic, then $B = 0$, but A arbitrary is admissible.

- For $\lambda = 0$, $r^2 F'' + rF' = 0$, so that $(rF')' = 0$ for $r > 0$, giving $r = c + d\log r$, where c, $d \in \mathbb{R}$.

- We conclude that for $\lambda = 0$ there is a nontrivial, 2π-periodic, separable solution in $r > 0$ of the form

$$T_0 = A_0 + B_0 \log r,$$

  where $A_0 = cA$ and $B_0 = dA$ are real constants. This solution is independent of θ and called the cylindrically-symmetric solution of $(\star)$.

<u>$\lambda = \omega^2$ ($\omega > 0$ wlog)</u>

- If $G'' + \omega^2 G = 0$, then $G(\theta) = R\cos(\omega\theta + \Phi)$, where R, $\Phi \in \mathbb{R}$.

- If G is nontrivial, then $R \neq 0$ and G has prime period $2\pi/\omega$. Hence, G can only be nontrivial and 2π-periodic if there exists a positive integer n such that $n \cdot 2\pi/\omega = 2\pi$, *i.e.* $\omega = n$ for some positive integer n, which the graph of G would reveal to be a geometrically obvious result.

- In anticipation of the need to write the solution in the form of a Fourier series, it is better to write the resulting solution for $\omega = n$ in the form $G(\theta) = A\cos n\theta + B\sin n\theta$, where $A = R\cos\Phi$, $B = -R\sin\Phi$ are arbitrary real constants.

- If $\lambda = \omega^2 = n^2$, then we obtain for $F(r)$ Euler's ODE in the form

$$r^2F'' + rF' - n^2F = 0 \quad \text{for} \quad r > 0.$$

- As in Introductory Calculus, we derive the general solution of this ODE by making the change of variable $r = e^t$, $F(r) = W(t)$. By the chain rule,

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}F}{\mathrm{d}r},$$

so that

$$\frac{\mathrm{d}^2W}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right)\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) = r^2F'' + rF' = n^2F = n^2W.$$

Hence, $W = Ce^{nt} + De^{-nt}$, where C, $D \in \mathbb{R}$, and we conclude that the general solution for $F(r)$ is given by

$$F(r) = Cr^n + Dr^{-n}.$$

- We not that an alternative method is to seek a solution of the form $F(r) = r^\mu$ for which $\mu(\mu-1) + \mu - \mu^2 = 0$ so that $\mu^2 = n^2$ giving $\mu = \pm n$. The general solution above then follows by the theory of second-order linear ODEs.

- We conclude that for $\lambda = \omega^2$ there are a countably infinite set of nontrivial, 2π-periodic, separable solution in $r > 0$ given for positive integers n by

$$T_n = (A_nr^b + B_nr^{-n})\cos n\theta + (C_nr^n + D_nr^{-n})\sin n\theta,$$

where $A_n = AC$, $B_n = AD$, $C_n = BC$, $D_n = BD$ are real constants.

- Superimposing the nontrivial, 2π-periodic, separable solutions in $r > 0$, we obtain the general series solution

$$T(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\Big((A_nr^n + B_nr^{-n})\cos n\theta + (C_nr^n + D_nr^{-n})\sin n\theta\Big). \qquad (\star\star)$$

Remarks

(1) We note that the solutions $\log r$, $r^{-n}\cos n\theta$ and $r^{-n}\sin n\theta$ are unbounded as $r \to 0+$, and hence not defined at $r = 0$. This means that these solutions are not admissible if the origin belongs to the domain in which T is defined.

(2) Similarly, if the domain in which T is defined extends to infinity and T is bounded there, then the solutions $\log r$, $r^n\cos n\theta$ and $r^n\sin n\theta$ are not admissible. We illustrate these results below with some concrete examples.

Example 1

- Consider the boundary value problem for T given by

$$\nabla^2 T = 0 \quad \text{in} \quad a < r < b, \tag{1}$$

  with

$$T = T_0^\star \quad \text{on} \quad r = a, \qquad T = T_1^\star \quad \text{on} \quad r = b, \tag{2}$$

  where a and b are constant radii, while $T_0^\star$ and $T_1^\star$ are constant temperatures.

- It follows from (1) that the general series solution ($\star\star$) pertains, so that the boundary conditions (2) can only be satisfied if

$$T_0^\star = A_0 + B_0 \log a + \sum_{n=1}^{\infty} \left((A_n a^n + B_n a^{-n}) \cos n\theta + (C_n a^n + D_n a^{-n}) \sin n\theta \right),$$
$$T_1^\star = A_0 + B_0 \log b + \sum_{n=1}^{\infty} \left((A_n b^n + B_n b^{-n}) \cos n\theta + (C_n b^n + D_n b^{-n}) \sin n\theta \right),$$

  for $-\pi < \theta \leq \pi$, say.

- Since the Fourier coefficients of a Fourier series are unique, we can equate them on the left- and right-hand sides of these equalities to obtain, for positive integers n,

$$\begin{bmatrix} 1 & \log a \\ 1 & \log b \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix}, \qquad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

  giving, since $a < b$,

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \frac{1}{\log\left(\frac{b}{a}\right)} \begin{bmatrix} \log b & -\log a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix}, \qquad \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

  and hence the cylindrically-symmetric solution

$$T = \frac{T_0^\star \log b - T_1^\star \log a}{\log\left(\frac{b}{a}\right)} + \frac{T_1^\star - T_0^\star}{\log\left(\frac{b}{a}\right)} \log r.$$

Remarks

(1) We note that the solution may be written in the form

$$\frac{T}{T_0^\star} = \frac{\log\left(\frac{r}{b}\right)}{\log\left(\frac{a}{b}\right)} + \frac{T_1^\star}{T_1^\star} \frac{\log\left(\frac{r}{b}\right)}{\log\left(\frac{b}{a}\right)}.$$

Since all of the fractions in this expression are dimensionless, we have verified that the solution is dimensionally correct.

(2) We note that we could have sought a circularly-symmetric solution $T = T(r)$ from the outset because the boundary data is independent of θ. However, the method above generalises to $T_0^\star$ and $T_1^\star$ being functions of θ.

Example 2

- Consider the boundary value problem for T given by

$$\nabla^2 T = 0 \quad \text{in} \quad r < a, \tag{1}$$

with

$$T(a,\theta) = T^\star \sin^3\theta \quad \text{for} \quad -\pi < \theta \le \pi, \tag{2}$$

where a is a constant radius and $T^\star$ is a constant temperature.

- Since T satisfies Laplace's equation in $r < a$, it must be twice differentiable with respect to x and y in a neighbourhood of the origin, and hence continuous and bounded at the origin, so that the general series solution ($\star\star$) pertains, but with $B_0 = 0$ and $B_n = D_n = 0$ for positive integers n.

- The boundary condition (2) can then only be satisfied if

$$T^\star \sin^3\theta = A_n + \sum_{n=1}^{\infty} (A_n a^n \cos n\theta + B_n a^n \sin n\theta) \quad \text{for} \quad -\pi < \theta \le \pi.$$

Since the Fourier seris for the left-hand side of this expression is given by the identity

$$T^\star \sin^3\theta = \frac{3T^\star}{4}\sin\theta - \frac{T^\star}{4}\sin 3\theta,$$

we can equate Fourier coefficients to deduce that

$$B_1 a = \frac{3T^\star}{4}, \; B_3 a^3 = -\frac{T^\star}{4}$$

while the remainder vanish, giving the solution

$$T = \frac{3T^\star}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{T^\star}{4}\left(\frac{r}{a}\right)^3 \sin 3\theta.$$

Remarks

(1) Question: What is the heat flux out of the disc through $r = a$? Answer: The heat flux vector $\boldsymbol{q} = -k\boldsymbol{\nabla} T$ according to Fourier's Law and we need the component in the direction of the outward pointing unit normal $\boldsymbol{n} = \boldsymbol{e}_r$ to the boundary, namely

$$\boldsymbol{q}\cdot\boldsymbol{n}|_{r=a} = (-k\nabla T)\cdot \boldsymbol{e}_r|_{r=a} = -k\frac{\partial T}{\partial r}(a,\theta) = -k\left(\frac{3T^\star}{4a}\sin\theta - \frac{3T^\star}{4a}\sin 3\theta\right).$$

(2) Since $\nabla^2 T = 0$ in $r < a$, we have $\nabla\cdot\boldsymbol{q} = 0$ in $r < a$, so that an application of the Divergence theorem in the plane gives

$$\int_{r=a} \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}s = \iint_{r<a} \nabla\cdot\boldsymbol{q}\,\mathrm{d}x\,\mathrm{d}y = 0,$$

i.e. the net flux of thermal energy through $r = a$ is equal to zero because there is no volumetric source or sink of thermal energy.