Fourier Series & PDEs: Lectures 15-16

Example 3: Poisson's Integral Formula

- Consider the boundary value problem for T given by

$$\nabla^2 T = 0 \quad \text{in} \quad r < a, \tag{1}$$

with

$$T(a,\theta) = f(\theta) \quad \text{for} \quad -\pi < \theta \leq \pi, \tag{2}$$

where a is a constant radius and the temperature profile f is given.

- As in Example 2, the general series solution of (1) is given by ($\star\star$) of last lecture, but with $B_0 = 0$ and $B_n = D_n = 0$ for positive integers n, *i.e.*

$$T(r,\theta) = A_0 + \sum_{n=1}^{\infty} \Big(A_n r^n \cos n\theta + C_n r^n \sin n\theta \Big).$$

- The boundary condition (2) can then only be satisfied if

$$f(\phi) = A_0 + \sum_{n=1}^{\infty} \left(A_n a^n \cos(n\phi) + B_n a^n \sin(n\phi) \right) \text{ for } -\pi < \phi \leq \pi,$$

where we replaced the dummy variable θ with ϕ in anticipation of the following analysis.

- The theory of Fourier series then gives the Fourier coefficients

$$2A_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi)\, \mathrm{d}\phi,$$

$$a^n A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \cos(n\phi)\, \mathrm{d}\phi,$$

$$a^n B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\phi) \sin(n\phi)\, \mathrm{d}\phi,$$

where n is a positive integer.

- While these integral expressions can evaluated in simple cases (such as in Example 2), it is a remarkable fact that the series solution may be summed for a wide class of functions f (namely those that are sufficiently regular that the following analysis is valid).

- We being by substituting the Fourier coefficients into the series solution and assuming that the orders of summation and integration may be interchanged to obtain

$$\begin{aligned} T(r,\theta) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\phi)\, \mathrm{d}\phi + \sum_{n=1}^{\infty} \left(\frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{r}{a}\right)^n [\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)] f(\phi)\, \mathrm{d}\phi \right) \\ &= \frac{1}{\pi} \int_{-\pi}^{\pi} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n(\theta - \phi) \right) f(\phi)\, \mathrm{d}\phi\,. \end{aligned}$$

- Now, if we let $\alpha = \theta - \phi$ and $z = \frac{r}{a}e^{i\alpha}$, then

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n\alpha &= \Re\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in\alpha}\right) \\
&= \Re\left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right) \\
&= \Re\left(\frac{1}{2} + \frac{z}{1-z}\right) \\
&= \frac{1}{2}\Re\left(\frac{1+z}{1-z}\right) \\
&= \frac{a^2 - r^2}{2(a^2 - 2ar\cos\alpha + r^2)},
\end{aligned}$$

where the summation of the geometric series in the third equality is valid for $|z| < 1$ and in the final equality we substituted for z and took the real part.

- Hence, we have derived Poisson's Integral Formula in the form

$$T(r,\theta) = \frac{(a^2 - r^2)}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)\,\mathrm{d}\phi}{a^2 - 2ar\cos(\theta - \phi) + r^2},$$

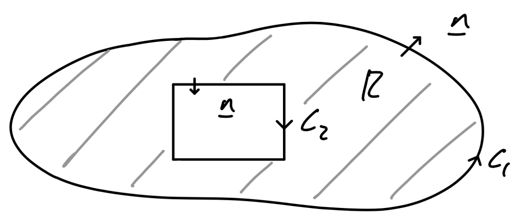
which is valid for $r < a$.

Uniqueness

- Uniqueness is established with the aid of the following version of Green's Theorem established in Multivariable Calculus.

- **Green's Theorem in the plane (Divergence Theorem Form)**

  Let R be a closed bounded region in the (x, y)-plane whose boundary ∂R is the union $C_1 \cup C_2 \cup \cdots \cup C_m$ of a finite number of piecewise-smooth simple closed curves (as illustrated in the figure).



Let $\boldsymbol{F} = (F_1(x, y), F_2(x, y))$ be continuous and have continuous first order derivatives on $R \cup \partial R$. Then

$$\iint_R \boldsymbol{\nabla} \cdot \boldsymbol{F}\,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} \boldsymbol{F} \cdot \boldsymbol{n}\,\mathrm{d}s\,,$$

where $\boldsymbol{n}$ is the outward pointing unit normal to ∂R in the (x, y)-plane and $\mathrm{d}s$ an element of arclength.

Uniqueness for the Dirichlet problem

- **Theorem:** Consider the Dirichlet problem for $T(x, y)$ given by

$$\boldsymbol{\nabla}^2 T = 0 \quad \text{in} \quad R,$$

with

$$T = f \quad \text{on} \quad \partial R,$$

where R is a bounded and path-connected region as in the statement of Green's Theorem in the plane and f is a given function. Then the boundary value problem has at most one solution.

- **Proof:** Let W be the difference between two solutions, then linearity gives

$$\boldsymbol{\nabla}^2 W = 0 \quad \text{in} \quad R,$$

with

$$W = 0 \quad \text{on} \quad \partial R,$$

The trick is to apply Green's theorem in the plane with $\boldsymbol{F} = W\boldsymbol{\nabla}W$ to obtain the integral identify

$$\iint_R \boldsymbol{\nabla} \cdot (W\boldsymbol{\nabla}W)\, \mathrm{d}x\, \mathrm{d}y = \int_{\partial R} (W\boldsymbol{\nabla}W) \cdot \boldsymbol{n}\, \mathrm{d}s.$$

Now $\boldsymbol{\nabla}^2 W = 0$ in R, so

$$\boldsymbol{\nabla} \cdot (W\boldsymbol{\nabla}W) = W\boldsymbol{\nabla}^2 W + \boldsymbol{\nabla}W \cdot \boldsymbol{\nabla}W = |\boldsymbol{\nabla}W|^2 \quad \text{in} \quad R,$$

while $W = 0$ on ∂R, so

$$W\boldsymbol{\nabla}W \cdot \boldsymbol{n} = 0 \text{ on } \partial R,$$

so the integral identity implies that

$$\iint_R |\boldsymbol{\nabla}W|^2\, \mathrm{d}x\, \mathrm{d}y = 0.$$

Assuming $\boldsymbol{\nabla}W$ is continuous on $R \cup \partial R$, we deduce that $\boldsymbol{\nabla}W = \boldsymbol{0}$ on R, so that W is constant on R because R is path connected. But $W = 0$ on ∂R, so assuming W is continuous on $R \cup \partial R$, the constant must vanish, and we deduce that $W = 0$ on $R \cup \partial R$, which completes the proof.

Example 1

- Find T such that $\boldsymbol{\nabla}^2 T = 0$ in $r < a$ with $T = T^\star x/a$ on $r = a$, where a and $T^\star$ are constants.
- If we can find any solution, then the uniqueness theorem guarantees it is the only solution.
- We could proceed via Fourier's method or Poisson's Integral Formula, but it is quicker to spot that the solution is simply $T = T^\star x/a$!

Example 2

- Find T such that $\boldsymbol{\nabla}^2 T = 0$ in $r > a$ with $T = T^\star x/a$ on $r = a$ and T bounded as $r \to \infty$.
- We can now spot that $T = B_1 r^{-1} \cos\theta$ is a solution provided $B_1 a^{-1} = T^\star$, but is this the only solution?

- The uniqueness theorem above is not applicable because R is not bounded. However, if W is the difference between two solutions, then for fixed $b > a$

$$\begin{aligned}\iint\limits_{a<r<b} |\boldsymbol{\nabla} W|^2 \, \mathrm{d}x \, \mathrm{d}y &= \iint\limits_{a<r<b} \boldsymbol{\nabla} \cdot (W \boldsymbol{\nabla} W) \, \mathrm{d}x \, \mathrm{d}y \\ &= \int\limits_{r=b} W \boldsymbol{\nabla} W \cdot \boldsymbol{e}_r \, \mathrm{d}s - \int\limits_{r=a} W \boldsymbol{\nabla} W \cdot \boldsymbol{e}_r \, \mathrm{d}s \\ &= \int\limits_{r=b} r W \frac{\partial W}{\partial r} \, \mathrm{d}s,\end{aligned}$$

where we used the same manipulations as in the proof of the uniqueness theorem above. If W decays sufficiently rapidly that $rW\frac{\partial W}{\partial r} \to 0$ as $r \to \infty$, then we can take the limit $b \to \infty$ in the last integral to deduce that

$$\iint\limits_{r>a} |\boldsymbol{\nabla} W|^2 \, \mathrm{d}x \, \mathrm{d}y = 0,$$

from which we may deduce uniqueness as before. It is beyond the scope of this course to show that if T and hence W is bounded at infinity, then $rW\frac{\partial W}{\partial r} \to 0$ as $r \to \infty$, though this may seem obvious from the general series solution.

Uniqueness for the Neumann problem

- **Theorem:** Consider the Neumann problem for $T(x, y)$ given by

$$\boldsymbol{\nabla}^2 T = f \quad \text{in} \quad R,$$

with

$$\frac{\partial T}{\partial n} = g \quad \text{on} \quad \partial R,$$

where R is a bounded and path-connected region as in the statement of Green's Theorem, $\partial T/\partial n \equiv \boldsymbol{n} \cdot \boldsymbol{\nabla} T$ is the outward normal derivative of T on ∂R, f is a given function on R and g is a given function on ∂R. Then the boundary value problem has no solution unless f and g satisfy the solvability condition

$$\iint\limits_R f \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial R} g \, \mathrm{d}s.$$

When a solution exists, it is not unique: any two solutions differ by a constant.

- **Proof:** Suppose there is a solution T, then

$$\iint\limits_R f \, \mathrm{d}x \, \mathrm{d}y = \iint\limits_R \boldsymbol{\nabla}^2 T \, \mathrm{d}x \, \mathrm{d}y = \iint\limits_R \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} T) \, \mathrm{d}x \, \mathrm{d}y = \int\limits_{\partial R} \boldsymbol{\nabla} T \cdot \boldsymbol{n} \, \mathrm{d}s = \int\limits_{\partial R} \frac{\partial T}{\partial n} \, \mathrm{d}s = \int\limits_{\partial R} g \, \mathrm{d}s,$$

where we used Green's Theorem with $\boldsymbol{F} = \boldsymbol{\nabla} T$ in the second equality.

Now let W be the difference between two solutions, so that linearity gives

$$\boldsymbol{\nabla}^2 W = 0 \quad \text{in} \quad R,$$

with

$$\frac{\partial W}{\partial n} = 0 \quad \text{on} \quad \partial R.$$

Then, as in the uniqueness proof for the Dirichlet problem,

$$\begin{aligned}
\iint_R |\boldsymbol{\nabla} W|^2 \, \mathrm{d}x \, \mathrm{d}y &= \iint_R \boldsymbol{\nabla} \cdot (W \boldsymbol{\nabla} W) \, \mathrm{d}x \, \mathrm{d}y \\
&= \int_{\partial R} W \boldsymbol{\nabla} W \cdot \boldsymbol{n} \, \mathrm{d}s \\
&= \int_{\partial R} W \frac{\partial W}{\partial n} \, \mathrm{d}s \\
&= 0.
\end{aligned}$$

Assuming $\boldsymbol{\nabla} W$ is continuous on $R \cup \partial R$, we dedude that $\boldsymbol{\nabla} W = \mathbf{0}$ on R, so that $W =$ constant on R. Hence, $W =$ constant on $R \cup \partial R$, assuming W is continuous there, *i.e.* any two solutions differ by a constant, which completes the proof.

Example 3

- Find T such that $\boldsymbol{\nabla}^2 T = 0$ in $r < a$ with $\partial T / \partial n = g(\theta)$ given on $r = a$.
- The general series solution of $\boldsymbol{\nabla}^2 T = 0$ in $r < a$ is given by

$$T = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos n\theta + B_n r^n \sin n\theta \right) .$$

so the boundary condition on $r = a$ can be satisfied only if

$$g(\theta) = \sum_{n=1}^{\infty} \left(n A_n a^{n-1} \cos n\theta + n B_n a^{n-1} \sin n\theta \right) \quad \text{for} \quad -\pi < \theta \leq \pi.$$

- The theory of Fourier series then requires

$$0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \, \mathrm{d}\theta \qquad (\dagger)$$

$$n A_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos n\theta \, \mathrm{d}\theta \qquad (n \in \mathbb{N} \setminus \{0\})$$

$$n B_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, \mathrm{d}\theta \, . \qquad (n \in \mathbb{N} \setminus \{0\})$$

- We deduce that there are two cases:

  (i) g is such that $(\dagger)$ is not true, in which case there is no solution;

  (ii) g is such that $(\dagger)$ is true, in which case the solution is not unique (since A_0 is arbitrary, while the other Fourier coefficients are uniquely determined).

- This agrees with the uniqueness theorem, which guarantees that in case (ii) we've found all possible solutions.

Well-posedness

- PDE problems often arise from modelling a particular physical system. In this case we could like to be able to make predictions as to the behaviour of the system based on our analysis of the PDE under consideration.
- **Definition:** A problem is said to be well-posed if the following three conditions are satisfied:
  - (I) EXISTENCE — there is a solution;
  - (II) UNIQUENESS — there is no more than one solution;
  - (III) CONTINUOUS DEPENDENCE — the solution depends continuously on the data.

Example: the wave equation

- Consider the initial value problem for $y(x,t)$ given by
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty,\ t > 0,$$
with the initial conditions
$$y(x,0) = f(x), \quad \frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for} \quad -\infty < x < \infty,$$
where the initial transverse displacement $f(x)$ and the initial transverse velocity $g(x)$ are given.
- By D'Alembert's Formula there exists a unique solution given by
$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$
Thus, (I) and (II) hold.
- Suppose we are interested in making predictions in the time interval $0 < t < T$ for some time T. Consider a similar problem
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad -\infty < x < \infty,\ t > 0,$$
$$y(x,0) = F(x), \quad \frac{\partial y}{\partial t}(x,0) = G(x), \quad -\infty < x < \infty,$$
where F and G are different initial data. Again, we know that there is exactly one solution:
$$Y(x,t) = \frac{1}{2}\big(F(x-ct) + F(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(s)\,\mathrm{d}s,$$
and
$$\begin{aligned} Y(x,t) - y(x,t) &= \frac{1}{2}\big((F(x-ct) - f(x-ct)) + (F(x+ct) - f(x+ct))\big) \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} \big(G(s) - g(s)\big)\,\mathrm{d}s. \end{aligned}$$
- Now let $\epsilon > 0$ be arbitrary and suppose that
$$|F(x) - f(x)| < \delta \text{ and } |G(x) - g(x)| < \delta \text{ for } -\infty < x < \infty.$$

Then

$$\begin{aligned}
|Y(x,t)-y(x,t)| &\le \frac{1}{2}|F(x-ct)-f(x-ct)| \\
&\quad +\frac{1}{2}|F(x+ct)-f(x+ct)| \\
&\quad +\frac{1}{2c}\int_{x-ct}^{x+ct}|G(s)-g(s)|\,\mathrm{d}s, \\
&< \frac{1}{2}\delta+\frac{1}{2}\delta+\frac{1}{2c}\int_{x-ct}^{x+ct}\delta\,\mathrm{d}s, \\
&= \frac{1}{2}\delta+\frac{1}{2}\delta+\frac{1}{2c}\cdot 2ct\delta, \\
&= (1+t)\delta \\
&< (1+T)\delta.
\end{aligned}$$

- Thus if the new data (F,G) are close to the original data (f,g) in the sense that

$$|F(x)-f(x)| < \frac{\epsilon}{1+T} \quad \text{and} \quad |G(x)-g(x)| < \frac{\epsilon}{1+T} \quad \text{for } -\infty < x < \infty,$$

then the corresponding solutions are close together in the sense that

$$|Y(x,t)-y(x,t)| < \epsilon \ \text{ for } -\infty < x < \infty \text{ and } 0 < t < T.$$

In this sense (III) holds and we conclude that the initial value problem for the wave equation is well-posed.

Example: IVP for Laplace's equation

- By contrast the corresponding initial value problem for Laplace's equation is not well-posed.
- Suppose we let $y(x,t)=0$, $f(x)=0$ and $g(x)=0$. Then y is a solution of the initial value problem

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 y}{\partial t^2}=0, \quad -\infty < x < \infty,\ t>0,$$

$$y(x,0)=f(x), \quad \frac{\partial y}{\partial t}(x,0)=g(x), \quad -\infty < x < \infty.$$

- Suppose now

$$Y(x,t)=\delta^2\cos\left(\frac{x}{\delta}\right)\sinh\left(\frac{t}{\delta}\right),\ F(x)=0,\ G(x)=\delta\cos\left(\frac{x}{\delta}\right).$$

Then $Y(x,t)$ is a solution of the initial value problem

$$\frac{\partial^2 Y}{\partial x^2}+\frac{\partial^2 Y}{\partial t^2}=0, \quad -\infty < x < \infty,\ t>0,$$

$$Y(x,0)=F(x), \quad \frac{\partial Y}{\partial t}(x,0)=G(x), \quad -\infty < x < \infty.$$

- Again suppose we want to make predictions in $0<t<T$. Then

$$|F(x)-f(x)|=0<\delta,\ \ |G(x)-g(x)|=\delta\left|\cos\left(\frac{x}{\delta}\right)\right|<\delta,$$

and

$$|Y(0,t)-y(0,t)|=\delta^2\sinh\left(\frac{t}{\delta}\right)<\delta^2\sinh\left(\frac{T}{\delta}\right).$$

- But
$$\delta^2 \sinh\left(\frac{T}{\delta}\right) = \frac{1}{2}\delta^2\left(e^{T/\delta} - e^{-T/\delta}\right) \to \infty \quad \text{as} \quad \delta \to 0,$$
and we cannot make
$$|Y(0,t) - y(0,t)| < \epsilon \quad \text{for} \quad 0 < t < T,$$
by making δ suitably small.
- The initial value problem for Laplace's equation is said to be ill-posed.

Summary

(1) Introduced theory of Fourier Series

- Periodic, even and odd functions and periodic extensions.
- Euler's formulae for Fourier coeffs via orthogonality relations.
- Statement of a powerful pointwise convergence theorem.
- Related rate of convergence to smoothness.
- Discussed Gibb's phenomenon - try to avoid!

(2) Heat equation

- Derivation in 1D.
- Simple solutions.
- Units and nondimensionalisation.
- Fourier's method for IBVPs.
- Generalised to inhomogeneous heat equation and BCs.
- Uniqueness

(3) Wave equation

- Derivation in 1D with gravity and air resistance.
- Normal modes and natural frequencies.
- Fourier's method for IBVPs - plucked and flicked strings.
- Forced and dampled wave equation with inhomogeneous BCs.
- Normal modes for weighted strings.
- D'Alembert's solution and characteristic diagrams.
- Uniqueness.

(4) Laplace's equation

- Fourier's method for BVPs in (x, y) and (r, θ).
- Poisson's Integral Formula for Dirichlet problem on a disk.
- Uniqueness of Dirichlet problem.
- Nonexistence and nonuniqueness of Neumann problem.

(5) Well-posedness

- Introduced concepts developed later on in course.