Fourier series & PDEs

Motivation

Example: existence of a convergent Fourier series

- Recall $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ for $z \in \mathbb{C}$.
- If we let $z = e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$, then

$$\mathrm{Im}(e^z) = \mathrm{Im}(e^{\cos\theta} e^{i\sin\theta}) = e^{\cos\theta}\sin(\sin\theta),$$

$$\mathrm{Im}(z^n) = \mathrm{Im}(e^{in\theta}) = \sin(n\theta).$$

- Hence, $e^{\cos\theta}\sin(\sin\theta) = \underbrace{\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n!}}_{\text{Fourier (sine) series}}$ for $\theta \in \mathbb{R}$.

Example: heat conduction

- Suppose $T(x,t)$ s.t. ① $T_t = T_{xx}$ for $0 < x < \pi$, $t > 0$
  ② $T(0,t) = 0$, $T(\pi,t) = 0$ for $t > 0$
  ③ $T(x,0) = e^{\cos x}\sin(\sin x)$ for $0 < x < \pi$.
- Observe $T(x,t) = \sum_{n=1}^{N} b_n \sin(nx) e^{-n^2 t}$ satisfies ① and ② for all $b_1, b_2, \dots, b_N \in \mathbb{R}$, $N \in \mathbb{N}\setminus\{0\}$.
- $Q^{\underline{n}}$: how should we pick N and the constants b_n?

  $A^{\underline{ns}}$: $N = \infty$ and $b_n = \frac{1}{n!}$ to satisfy ③, i.e. a sol$^{\underline{n}}$ of the IBVP ① - ③ is $T(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!}\sin(nx)e^{-n^2 t}$.

- But what about other initial conditions?

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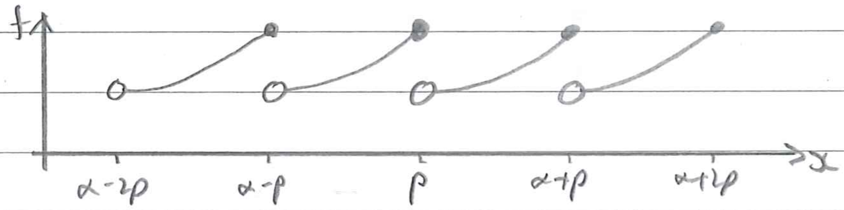
Periodic, even and odd functions

- **Defn**: $f:\mathbb{R}\to\mathbb{R}$ is a periodic function if $\exists p>0$ s.t. $f(x+p)=f(x)\ \forall x\in\mathbb{R}$. In this case p is a period for f and f is called p-periodic. Period is not unique, but if $\exists$ smallest such p, it is called the prime period.

- E.g. $f=$ const. is p-periodic $\forall p>0$, so has no prime period.
  $\sin x$ has prime period 2π.
  x & x^2 not periodic.

- Note graph of f repeats every p, e.g.



- $f:(\alpha,\alpha+p]\to\mathbb{R}$ can be extended uniquely to be p-periodic.

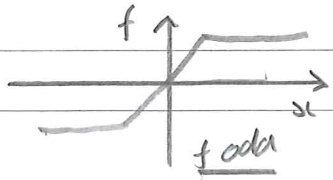
- **Defn**: The periodic extension $F:\mathbb{R}\to\mathbb{R}$ of $f:(\alpha,\alpha+p]\to\mathbb{R}$ is defined by $F(x)=f(x-mp)$, where for each $x\in\mathbb{R}$, m is the unique integer s.t. $x-mp\in(\alpha,\alpha+p]$.

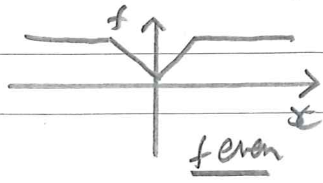
- f,g p-periodic $\Rightarrow$ (1) f,g np-periodic $\forall n\in\mathbb{N}\setminus\{0\}$
  (2) $\alpha f+\beta g$ p-periodic $\forall \alpha,\beta\in\mathbb{R}$
  (3) fg p-periodic
  (4) $f(\lambda x)$ p/λ-periodic $\forall \lambda>0$
  (5) $\int_0^p f(x)\,dx=\int_\alpha^{\alpha+p} f(x)\,dx\ \forall\alpha\in\mathbb{R}$

- **Defn**: $f:\mathbb{R}\to\mathbb{R}$ odd if $f(x)=-f(-x)\ \forall x\in\mathbb{R}$
  $f:\mathbb{R}\to\mathbb{R}$ even if $f(x)=f(x)\ \forall x\in\mathbb{R}$

- E.g. x^n odd for n odd, even for n even (hence names)

- Note symmetries of graphs:





- Properties of odd/even functions: If f, f_1 odd and g, g_1 even, then (1) $f(0) = 0$;
  (2) $\int_{-\alpha}^{\alpha} f(x)\,dx = 0 \quad \forall \alpha \in \mathbb{R}$;
  (3) $\int_{-\alpha}^{\alpha} g(x)\,dx = 2\int_0^{\alpha} g(x)\,dx \quad \forall x \in \mathbb{R}$;
  (4) fg odd, ff_1 even, gg_1 even.

Fourier series for functions of period 2π

- Let $f: \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π.
- We want an expansion for f of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)) \qquad (*)$$

- Q1: If $(*)$ is true, can we find the constants a_n, b_n in terms of f?

  Q2: With these a_n & b_n, when is $(*)$ true?

Question 1

- Suppose $(*)$ true and we can integrate it term by term, then

$$\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty}\Big(a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\,dx}_{=0} + b_n \underbrace{\int_{-\pi}^{\pi}\sin(nx)\,dx}_{=0}\Big)$$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,$$ i.e. $\frac{a_0}{2}$ is the mean of f over a period.

- Lemma: Let $m, n \in \mathbb{N}\setminus\{0\}$. Then we have the orthogonality relations:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = \pi\delta_{mn}, \quad \int_{-\pi}^{\pi}\cos(mx)\sin(nx)\,dx = 0, \quad \int_{-\pi}^{\pi}\sin(mx)\sin(nx)\,dx = \pi\delta_{mn},$$

where δ_{mn} is Kronecker's delta, i.e. $\delta_{mn} = \begin{cases} 1 & \text{for } m = n, \\ 0 & \text{for } m \neq n. \end{cases}$

Pf: see online notes and sheet 1.

④

- Fix $m \in \mathbb{N} \setminus \{0\}$, multiply (*) by $\cos(mx)$ and assume $\int \Sigma = \Sigma \int$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,dx + \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx + b_n \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,dx\right)$$

$$= \frac{1}{2}a_0 \cdot 0 + \sum_{n=1}^{\infty}\left(a_n \pi \delta_{mn} + b_n \cdot 0\right)$$

$$= \pi a_m$$

$$\Rightarrow \quad \underline{\underline{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx \quad \text{for } m \in \mathbb{N} \setminus \{0\}}}$$

- Similarly, we can fix $m \in \mathbb{N} \setminus \{0\}$, multiply (*) by $\sin(mx)$ and assume $\int \Sigma = \Sigma \int$ to get

$$\underline{\underline{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx \quad \text{for } m \in \mathbb{N} \setminus \{0\}}}$$

- <u>Defn</u>: Suppose f is such that the <u>Fourier coefficients</u>

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \ (n \in \mathbb{N}), \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \ (n \in \mathbb{N} \setminus \{0\})$$

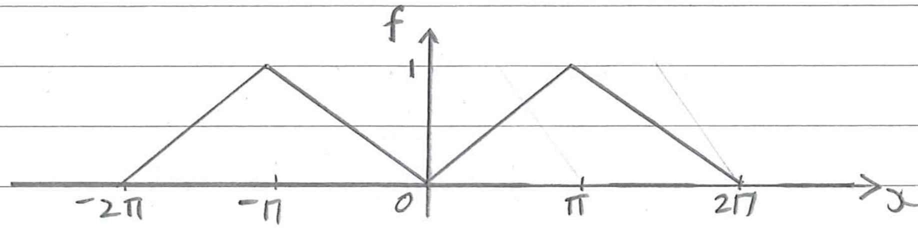
exist. Then we write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos(nx) + b_n\sin(nx)\right),$$

where $\sim$ means the RHS is the <u>Fourier series for f</u>, regardless of whether or not it converges to f.

- Note factor of $\frac{1}{2}$ in first term is for algebraic convenience.

- Example 2.1 : Find the Fourier series (FS) for the 2π-periodic function f defined by $f(x) = |x|$ for $-\pi < x \leq \pi$.



$f(x)$ even $\Rightarrow$ $f(x)\cos(nx)$ even and $f(x)\sin(nx)$ odd

$$\Rightarrow \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx, \quad b_n = 0$$

Calculate $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \left[\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi} = \pi.$

For $n > 0$, we use integration by parts:

$$(uv)' = u'v + uv' \Rightarrow \quad [uv]_a^b = \int_a^b u'v + uv'\,dx$$

Pick $u = x$, $v = \frac{1}{n}\sin(nx)$, $a = 0$, $b = \pi$

$$\Rightarrow \left[\frac{x}{n}\sin(nx)\right]_0^{\pi} = \int_0^{\pi} 1\cdot\frac{1}{n}\sin(nx) + x\cos(nx)\,dx$$

$$\Rightarrow \int_0^{\pi} x\cos(nx)\,dx = -\int_0^{\pi}\frac{1}{n}\sin(nx)\,dx = \left[\frac{\cos(nx)}{n^2}\right]_0^{\pi} = \frac{(-1)^n - 1}{n^2}$$

$$\Rightarrow a_n = -\frac{2(1-(-1)^n)}{\pi n^2} = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\} \\ -\frac{4}{\pi(2m+1)^2} & \text{for } n = 2m+1,\ m \in \mathbb{N} \end{cases}$$

Hence,

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2} \qquad \square$$

- Remarks

(1) Partial sums defined by

$$S_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{N}\frac{\cos((2m+1)x)}{(2m+1)^2} \quad \text{for } N \in \mathbb{N}.$$

Plots in handout suggest that FS converges on $\mathbb{R}$, i.e.

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \text{for } x \in \mathbb{R}. \qquad (+)$$

(2) If this is true, can pick x to evaluate the sum of a series, e.g. $x = 0$

$$\Rightarrow \quad 0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \Rightarrow \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Sine and cosine series

- Let f be 2π-periodic and s.t. Fourier coefficients exist.
- $f(x)$ odd $\Rightarrow$ $f(x)\cos(nx)$ odd and $f(x)\sin(nx)$ even

$$\Rightarrow \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$\Rightarrow \quad f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx),$$ called a <u>Fourier sine series</u>

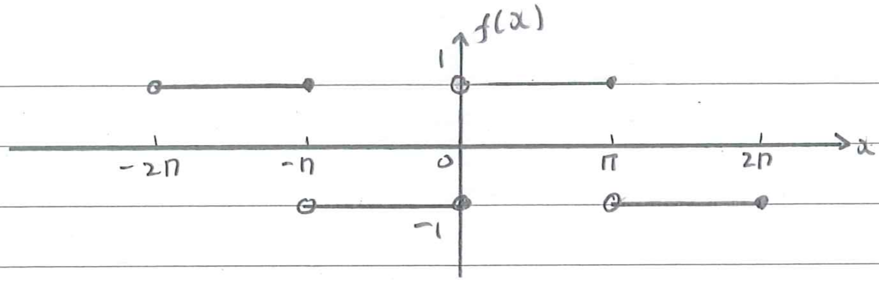
- Note this is also true if f is odd only for $x \neq k\pi$, $k \in \mathbb{Z}$.
- Similarly, f even $\Rightarrow f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$, called a <u>Fourier cosine series</u>, where $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$.

Question 2

- When does the FS for f converge?

<u>Example 2.2</u>: Find the FS for the 2π-periodic function f defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x \leq \pi, \\ -1 & \text{for } -\pi < x \leq 0. \end{cases}$$



f odd for $x \neq k\pi, k \in \mathbb{Z} \Rightarrow a_n = 0, \ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

$f(x) = 1$ for $0 < x < \pi \Rightarrow b_n = \left[\frac{-2}{\pi}\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2(1-(-1)^n)}{\pi n}$

$$\Rightarrow f(x) \sim \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin((2m+1)x)}{2m+1} \qquad \square$$

Remarks

(1) Partial sums defined by

$$S_N(x) = \frac{4}{\pi}\sum_{m=0}^{N}\frac{\sin((2m+1)x)}{2m+1} \quad \text{for } N \in \mathbb{N}.$$

Plots in handout suggest that

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x) & \text{for } x \neq k\pi, k \in \mathbb{Z}, \\ 0 & \text{for } x = k\pi, k \in \mathbb{Z}. \end{cases} \qquad (\#)$$

(2) Note slower convergence than in example 2.1 and persistent overshoot near discontinuities of f — called Gibb's phenomenon (more later).

Convergence of Fourier series

- **Def$^{\underline{n}}$:** $f(c_+) = \lim_{\substack{h\to 0 \\ h>0}} f(c+h)$ if it exists (RH limit at c)

  $f(c_-) = \lim_{\substack{h\to 0 \\ h<0}} f(c+h)$ if it exists (LH limit at c)

- Remarks:

(1) $f(c)$ need not be defined for $f(c+)$ or $f(c-)$ to exist.

(2) Existence part important, e.g. $f(x) = \sin(\frac{1}{x})$ for $x \neq 0 \Rightarrow$ $f(0\pm)$ do not exist.

(3) $f(c+) = f(c-) = f(c) \Leftrightarrow f$ continuous at c.

(4) In example 2.2, f is continuous for $x \neq k\pi$, $k \in \mathbb{Z}$, with e.g. $f(0+) = 1$, $f(0-) = -1$, $f(\pi+) = -1$, $f(\pi-) = 1$.

- <u>Defn</u>: f is <u>piecewise continuous</u> on $(a,b) \subseteq \mathbb{R}$ if there exist a finite number of points $x_1, x_2, \ldots, x_m$ with $a = x_1 < x_2 < \ldots < x_m = b$ such that
  (i) f is defined and continuous on (x_k, x_{k+1}) $\forall k = 1, \ldots, m-1$;
  (ii) $f(x_k+)$ exists for $k = 1, \ldots, m-1$;
  (iii) $f(x_k-)$ exists for $k = 2, \ldots, m$.

- Note that f need not be defined at its exceptional points $x_1, \ldots, x_m$!

- E.g. the functions in examples 2.1 and 2.2 are piecewise continuous on any interval $(a,b) \subseteq \mathbb{R}$.

- <u>Fourier Convergence Theorem (FCT)</u>

Let f be 2π-periodic, with f and f' piecewise continuous on $(-\pi, \pi)$. Then, the Fourier coefficients a_n and b_n exist, and

$$\frac{1}{2}\left(f(x+) + f(x-)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

for $x \in \mathbb{R}$.

- Discuss further next time, but note here that FCT $\Rightarrow$ (+) and (++) are true.

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Remarks

(1) f, f' piecewise continuous (p.c.) on $(-\pi,\pi)$ $\Rightarrow$ $\exists\, x_1, x_2, \ldots, x_m \in \mathbb{R}$ with $-\pi = x_1 < x_2 < \ldots < x_m = \pi$ such that

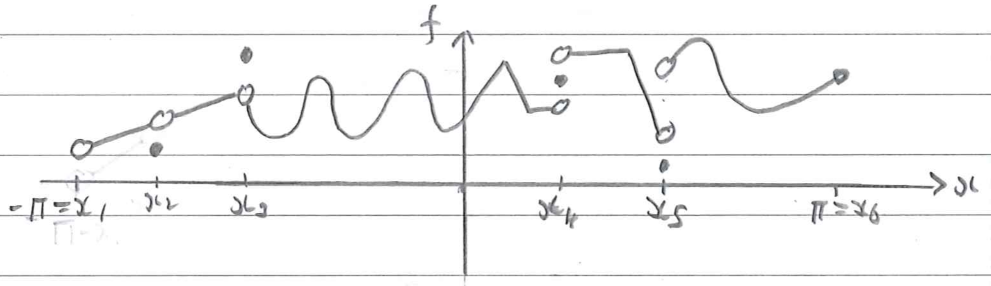
(i) f and f' are continuous on (x_k, x_{k+1}) for $k = 1, \ldots, m$;

(ii) $f(x_{k+})$ and $f'(x_{k+})$ exist for $k = 1, \ldots, m-1$;

(iii) $f(x_{k-})$ and $f'(x_{k-})$ exist for $k = 2, \ldots, m$.

Thus, in any period f, f' are continuous except possibly at a finite number of points; at each such point f' need not be defined, and one or both of f and f' may have a jump discontinuity.

E.g.



E.g. $f(x) = \begin{cases} x^{1/2} & \text{for } 0 < x \le \pi \\ 0 & \text{for } -\pi < x \le 0 \end{cases} \Rightarrow f'(x) = \begin{cases} \frac{1}{2} x^{-1/2} & \text{for } 0 < x < \pi \\ 0 & \text{for } -\pi < x < 0 \\ \text{undefined} & \text{for } x = 0, \pi \end{cases}$

$\Rightarrow f$ p.c. on $(-\pi, \pi)$, but f' not.

(2) Proof not examinable, but one method is as follows: firstly, show that

$$\frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos(nx) + b_n \sin(nx)\right) - \frac{1}{2}\left(f(x_+) + f(x_-)\right) = \int_0^{\pi} F(x,t) \sin\left((N+\tfrac{1}{2})t\right) dt,$$

where $F(x,t) = \frac{1}{\pi}\left(\frac{f(x+t) - f(x_+)}{t} + \frac{f(x-t) - f(x_-)}{t}\right)\left(\frac{t}{2\sin(t/2)}\right)$;

secondly, show $F(x,t)$ is a p.c. f^n of t on $(0,\pi)$, so that Riemann-Lebesgue Lemma (Analysis III) implies

$$\int_0^{\pi} F(x,t) \sin\left((N+\tfrac{1}{2})t\right) dt \to 0 \text{ as } N \to \infty.$$

(10)

(3) f continuous at $x \Rightarrow \frac{1}{2}(f(x_+) + f(x_-)) = f(x)$.

(4) If f defined only on e.g. $(-\pi, \pi]$, FCT holds for its 2π-periodic extension.

(5) Can integrate termwise under weaker conditions, e.g. if f is only 2π-periodic and p.c. on $(-\pi, \pi)$, then FCT

$$\Rightarrow \int_0^X f(x)\,dx = \frac{1}{2}a_0 X + \sum_{n=1}^{\infty}\left(a_n \int_0^X \cos(nx)\,dx + b_n \int_0^X \sin(nx)\,dx\right)$$

for $X \in \mathbb{R}$; note that LHS is 2π-periodic iff $a_0 = 0$.

(6) But need stronger conditions to differentiate termwise, e.g. if f is 2π-periodic and continuous on $\mathbb{R}$ with f' and f'' p.c. on $(-\pi, \pi)$, then FCT

$$\Rightarrow \frac{1}{2}(f'(x_+) + f'(x_-)) = \sum_{n=1}^{\infty}\left(a_n \frac{d}{dx}(\cos(nx)) + b_n \frac{d}{dx}(\sin(nx))\right)$$

for $x \in \mathbb{R}$.

Rate of convergence

- The smoother f, i.e. the more continuous derivatives, the faster the convergence of the FS for f.

- If the first jump discontinuity is in the pth derivative of f, with the convension that $p = 0$ if there is a jump discontinuity in f, then typically the non-zero a_n and b_n decay like $\frac{1}{n^{p+1}}$ as $n \to \infty$. E.g $p = 1$ in ex. 2.1, while $p = 0$ in ex. 2.2.

- Extremely useful result in practice (e.g. how many terms to keep for an accurate approximation) and for checking calculations. e.g. for $\approx 1\%$ accuracy need 100 terms for $p=1$, 10 for $p=0$.

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Gibb's phenomenon

- This is the persistent overshoot in ex. 2.2 near a jump discontinuity.
- Happens whenever $\exists$ a jump discontinuity.
- As # terms in partial sum $\to \infty$,

  width overshoot region $\to 0$ (by FCT)

  height overshoot region $\to \gamma |f(x_+)-f(x_-)|$,

  where $\gamma = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin x}{x}\,dx \simeq 1.18$, i.e. $\simeq 9\%$ (top & bottom.
- Awful for approximation purposes!

Functions of any period

- Suppose now $f(x)$ is a periodic function of period $2L>0$.
- Make transformation $x = \frac{LX}{\pi}$, $f(x) = g(X)$, then for $X \in \mathbb{R}$,

$$g(X+2\pi) \underset{\substack{\uparrow \\ (g(X)=f(\frac{L}{\pi}X))}}{=} f\left(\tfrac{L}{\pi}(X+2\pi)\right) = f\left(\tfrac{LX}{2\pi}+2L\right) \underset{\substack{\uparrow \\ (f\ 2L\text{-periodic})}}{=} f\left(\tfrac{LX}{\pi}\right) \underset{\uparrow}{=} g(X).$$

Thus, g is 2π-periodic and we can use transformation to derive theory for f from that for g above.

- Here we summarize the key results.
- FS: $g(X) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos(nX) + b_n\sin(nX)\right)$

$$\Leftrightarrow f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right)$$

(12)

- <u>Fourier coefficients:</u>

$$a_n \underset{\text{(defn)}}{=} \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)\,dx \underset{(x=\frac{\pi x}{L})}{=} \frac{1}{\pi}\int_{-L}^{L} g\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{dx}{L} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Similarly, $b_n = \frac{1}{L}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$

- <u>Important remark</u>: These formulae may be derived directly from the FS for f by assuming $\int\sum = \sum\int$ and using the orthogonality relation

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx = L\,\delta_{mn},$$

$$\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0,$$

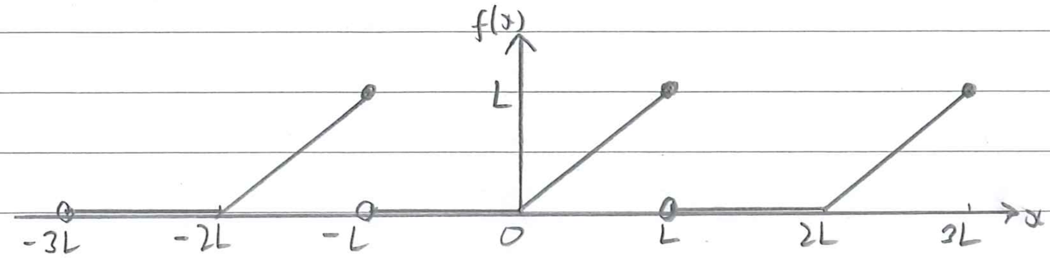
$$\int_{-L}^{L}\sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = L\,\delta_{mn},$$

where $n,m \in \mathbb{N}\setminus\{0\}$ and $\delta_{mn} = \begin{cases} 1 & \text{for } m=n, \\ 0 & \text{for } m\neq n. \end{cases}$

- <u>FCT</u>: Let f be $2L$-periodic with f and f' p.c. on $(-L,L)$. Then a_n and b_n exist, and

$$\frac{1}{2}\left(f(x_+) + \frac{1}{2}f(x_-)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right) \text{ for } x\in\mathbb{R}.$$

Example 2.3: Find the FS of the 2L-periodic function f defined by $f(x) = \begin{cases} x & \text{for } 0 < x \leq L, \\ 0 & \text{for } -L < x \leq 0. \end{cases}$



$$a_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx, \quad b_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

Find $a_0 = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$, but for $n > 0$ it is a bit quicker to evaluate

$$a_n + ib_n = \frac{1}{L}\int_0^L \underbrace{x}_{u}\ \underbrace{\exp\left(\frac{in\pi x}{L}\right)}_{v'} dx$$

$$= \left[\frac{1}{L}\underbrace{x}_{u}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\right]_0^L - \frac{1}{L}\int_0^L \underbrace{1}_{u'}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v} dx$$

$$= \frac{L}{in\pi}\exp(in\pi) - \left[\frac{1}{L}\left(\frac{L}{in\pi}\right)^2\exp\left(\frac{in\pi x}{L}\right)\right]_0^L$$

$$= \frac{iL(-1)^{n+1}}{n\pi} + \frac{L}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$\Rightarrow f(x) \sim \frac{L}{4} + \sum_{m=1}^{\infty}\left(-\frac{2L}{(2m-1)^2\pi^2}\cos\left(\frac{(2m-1)\pi x}{L}\right) + \frac{L(-1)^{m+1}}{m\pi}\sin\left(\frac{m\pi x}{L}\right)\right)$$

□

- FCT $\Rightarrow$ FS converges to $f(x)$ for $x \neq (2k+1)L$, $k \in \mathbb{Z}$, and to $\frac{1}{2}(f(L+) + f(L-)) = \frac{1}{2}(0 + L) = \frac{L}{2}$ otherwise.

- E.g. $x = 0 \Rightarrow 0 = f(0) = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2} \Rightarrow \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$

  $x = L \Rightarrow \frac{L}{2} = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)}{(2m-1)^2} \Rightarrow$ same sum!

Cosine and sine series

- Suppose now $f:[0,L]\to\mathbb{R}$ given. Periodic extension of period $2L$ not unique, but there are two especially useful ones for PDE applications.

- **Defn**: The even/odd $2L$-periodic extensions, f_e and f_o respectively, of $f:[0,L]\to\mathbb{R}$ are defined by
$$f_e(x)=\begin{cases} f(x) \text{ for } 0\le x\le L,\\ f(-x) \text{ for } -L<x<0,\end{cases} \quad \text{with } f_e(x+2L)=f_e(x) \text{ for } x\in\mathbb{R}$$
and
$$f_o(x)=\begin{cases} f(x) \text{ for } 0\le x\le L,\\ -f(-x) \text{ for } -L<x<0,\end{cases} \quad \text{with } f_o(x+2L)=f_o(x) \text{ for } x\in\mathbb{R}.$$
*→

- **Defn**: The Fourier cosine and sine series for $f:[0,L]\to\mathbb{R}$ are the Fourier series for f_e and f_o respectively, i.e.
$$f_e(x)\sim\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right), \text{ where } a_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx;$$
$$f_o(x)\sim\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right), \text{ where } b_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

- Note that if f is continuous on $[0,L]$ and f' p.c. on $[0,L)$, then FCT gives
$$\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{L}\right)=f_e(x) \text{ for } x\in\mathbb{R};$$
$$\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{L}\right)=\begin{cases} f_o(x) \text{ for } x\neq kL,\ k\in\mathbb{Z},\\ 0 \text{ otherwise.}\end{cases}$$

- **Example 2.4**: Find the cosine and sine series of $f:[0,L]\to\mathbb{R}$ defined by $f(x)=x$ for $0\le x\le L$.
$$f_e(x)=\begin{cases} x \text{ for } 0\le x\le L,\\ -x \text{ for } -L<x<0\end{cases}, \quad \text{i.e. } f_e(x)=|x| \text{ for } -L<x\le L$$

* Note that $f_o(x)$ is odd for $x\neq kL$, $k\in\mathbb{Z}$, and odd (on $\mathbb{R}$) iff $f(0)=f(L)=0$.

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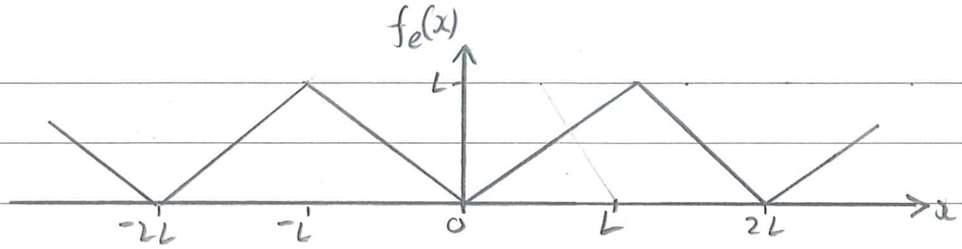
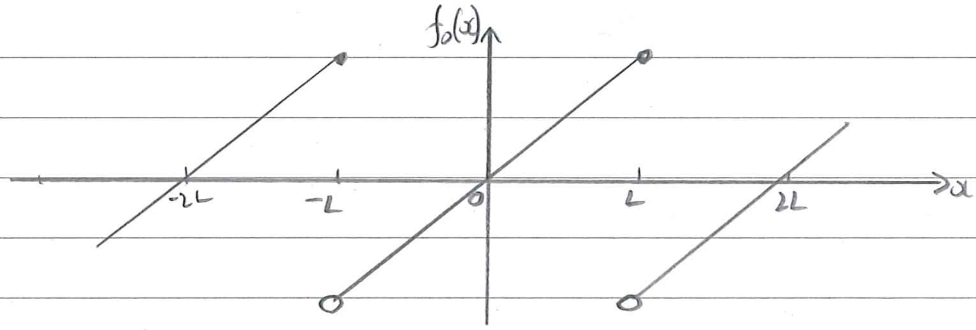


$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow f_e(x) \sim \underbrace{\frac{L}{2} - \sum_{m=0}^{\infty} \frac{4L}{(2m+1)^2\pi^2}\cos\left(\frac{(2m+1)\pi x}{L}\right)}_{\text{Cosine series}} \underset{\uparrow \text{FCT}}{=} f_e(x)$$

$$f_o(x) = \begin{cases} x & \text{for } 0 \le x \le L, \\ -(-x) & \text{for } -L < x < L \end{cases} \Rightarrow f_o(x) = x \text{ for } -L < x \le L$$



$$b_n = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow f_o(x) \sim \underbrace{\sum_{n=1}^{\infty} \frac{2L(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)}_{\text{Sine series}} \underset{\uparrow \text{FCT}}{=} \begin{cases} f_o(x) & \text{for } x \neq kL, k \in \mathbb{Z}, \\ 0 & \text{otherwise.} \end{cases}$$

□

<u>Remarks</u>

(1) $f_e + f_o = 2f_{ex.23} \Rightarrow FS(f_e) + FS(f_o) = FS(2f_{ex.22})$

(2) Rates of convergence? $p = 1$ for f_e ✓ $p = 0$ for f_o ✓

(3) <u>Qn</u>: Which truncated series gives best approx to f on $[0, L]$?
<u>Ans</u>: Cosine series since (i) it converges everywhere on $[0, L]$;
(ii) it converges more rapidly;
(iii) it does not exhibit Gibb's phenomenon.