The PDEs we shall study

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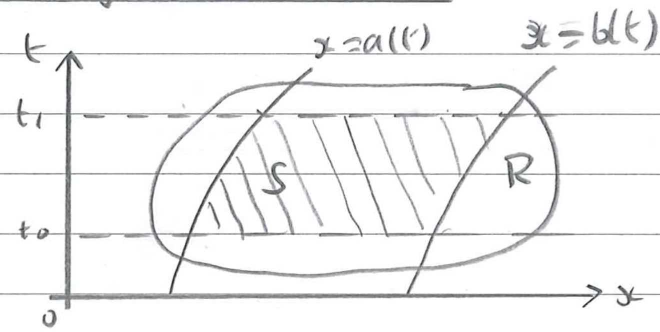
PDE	Name	Unknown	Parameters
$T_t = \kappa T_{xx}$	Heat equation	$T(x,t)$	$\kappa > 0$
$y_{tt} = c^2 y_{xx}$	Wave equation	$y(x,t)$	$c > 0$
$T_{xx} + T_{yy} = 0$	Laplace's equation	$T(x,y)$	None

- We shall derive them using physical principles and develop methods to solve several physically important problems formed by imposing appropriate BCs and/or ICs — different for each of them!

Some preliminaries

- Leibniz's Integral Rule (LIR)



If F, F_t are continuous on $R \supseteq S$ and $a, \dot{a}, b, \dot{b}$ are continuous for $t \in [t_0, t_1]$ then

$$\frac{d}{dt}\int_{a(t)}^{b(t)} F(x,t)\,dx = \int_{a(t)}^{b(t)} F_t(x,t)\,dx + F(b(t),t)\,\dot{b}(t) - F(a(t),t)\,\dot{a}(t).$$

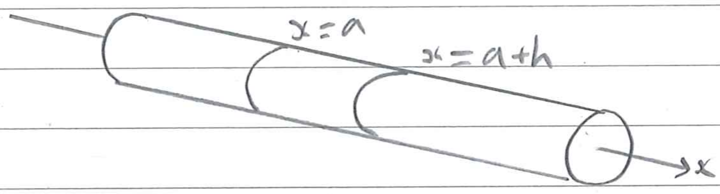
Note: a, b constant $\Rightarrow \frac{d}{dt}\int_a^b F(x,t)\,dx = \int_a^b F_t(x,t)\,dx$

- Lemma (1.2): $f(x)$ cts $\Rightarrow \frac{1}{h}\int_a^{a+h} f(x)\,dx \to f(a)$ as $h \to 0$.

The heat equation

Derivation in 1D

- Consider a straight rigid isotropic conducting rod (e.g. metal) with insulated lateral surfaces lying along x-axis.



- We'll need following quantities.

Symbol	Quantity	SI units
x	Axial distance	m
t	Time	s
$T(x,t)$	Temperature	K
$q(x,t)$	Heat flux in the x-direction	$J m^{-2} s^{-1}$ $(1J = 1Nm)$
A	Cross-sectional area	m^2
ρ	Rod density	$Kg\, m^{-3}$
c	Rod specific heat	$J\, Kg\, K^{-1}$
k	Rod thermal conductivity	$(J K^{-1} m^{-1} s^{-1})$
K	Rod thermal diffusivity	$(m^2 s^{-1})$

- Conservation of energy in fixed section $a \leq x \leq a+h$:

$$\underbrace{\frac{d}{dt}\left(A\int_a^{a+h} \rho c T \, dx\right)}_{1} = \underbrace{Aq(a,t)}_{2} \; - \underbrace{Aq(a+h,t)}_{3}.$$

1 is time rate of change of internal energy in $a \leq x \leq a+h$.

2 is rate at which heat enters through $x = a$.

3 is rate at which heat leaves through $x = a+h$.

- Note also true for $h < 0$ with appropriate reinterpretation.
- Assuming T_t is cts, LIR with $a, a+h$ constant gives

$$\frac{\rho c}{h}\int_a^{a+h} T_t \, dx + \frac{q(a+h,t) - q(a,t)}{h} = 0$$

- Assuming q_x is cts and taking limit $h \to 0$, Lemma (1.2) gives

$$\rho c T_t + q_x = 0. \qquad (+)$$

Fourier's law

- This is the constitutive law

$$q = -k T_x \qquad (\#)$$

- Models flow of heat from high to low temperatures.
- $(+)$ & $(\#)$ $\Rightarrow$ $\rho c T_t - (k T_x)_x = 0$ or $\boxed{T_t = \kappa T_{xx}}$ Heat equation
  where $\kappa = k/\rho c$.
- Note we assumed T_t and $q_x = -k T_{xx}$ to be cts.

Units and nondimensionalization

- Notation: $[p]$ = dimension of p in fundamental dimensions (M, L, T, Θ etc) or e.g. SI units (Kg, m, s, K etc).
- Both sides of an equation modelling a physical process must have same dimensions, e.g. $[①] = [②] = [③] = J s^{-1}$.
- Exploit to check solutions are dimensionally correct and to determine dimensions of parameters, e.g.

$$[k] = \frac{[q]}{[T_x]} = \frac{J m^{-2} s^{-1}}{K m^{-1}} = J K^{-1} m^{-1} s^{-1}, \quad \kappa = \frac{[T_t]}{[T_{xx}]} = \frac{[x^2]}{[t]} = m^2 s^{-1}.$$

- Nondimensionalization: Method of scaling variables with typical values to derive dimensionless equations. These usually contain dimensionless parameters that characterize the relative importance of the physical mechanisms in the model.

- E.g. IBVP

- Suppose $T(x,t)$ s.t. ① $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;
  ② $T(0,t) = T_0 + T_1 \sin(\Omega t)$, $T_x(L,t) = 0$ for $t > 0$;
  ③ $T(x,0) = T_2 \frac{x}{L}\left(1 - \frac{x}{L}\right)$ for $0 < x < L$.

- Six dimensional parameters: $\kappa, L, T_0, T_1, \Omega, T_2$.

- Nondimensionalise by scaling $x = L\hat{x}$, $t = \tau\hat{t}$, $T = T_2\hat{T}(\hat{x},\hat{t})$, where timescale τ is to be chosen.

- Chain rule $\Rightarrow \frac{\partial T}{\partial t} = T_2 \frac{\partial \hat{T}}{\partial \hat{t}}\frac{\partial \hat{t}}{\partial t} = \frac{T_2}{\tau}\frac{\partial \hat{T}}{\partial \hat{t}}$, $\frac{\partial T}{\partial x} = \frac{T_2}{L}\frac{\partial \hat{T}}{\partial \hat{x}}$ etc.

- Hence ①' $\frac{\partial \hat{T}}{\partial \hat{t}} = \frac{\kappa\tau}{L^2}\frac{\partial^2 \hat{T}}{\partial \hat{x}^2}$ for $0 < \hat{x} < 1$, $\hat{t} > 0$;

  ②' $\hat{T}(0,\hat{t}) = \frac{T_0}{T_2} + \frac{T_1}{T_2}\sin(\Omega\tau\hat{t})$, $\hat{T}_{\hat{x}}(1,\hat{t}) = 0$ for $\hat{t} > 0$;

  ③' $\hat{T}(\hat{x},0) = \hat{x}(1-\hat{x})$ for $0 < \hat{x} < 1$.

- Choose $\tau = \frac{L^2}{\kappa}$, i.e. timescale for diffusive transport of heat, and drop hats: ①'' $T_t = T_{xx}$ for $0 < x < 1$, $t > 0$;
  ②'' $T(0,t) = \alpha_1 + \alpha_2 \sin(\omega t)$, $T_x(1,t) = 0$ for $t > 0$;
  ③'' $T(x,0) = x(1-x)$ for $0 < x < 1$.

- Three dimensionless parameters: $\alpha_1 = \frac{T_0}{T_2}$, $\alpha_2 = \frac{T_1}{T_2}$, $\omega = \Omega\tau = \frac{\Omega L^2}{\kappa}$.

- Hope to simplify if e.g. α_2 or ω is small.

Heat conduction in a finite rod

- Consider IBVP for $T(x,t)$ given by
  (1) $T_t = \nu_1 T_{xx}$ for $0 < x < L$, $t > 0$;
  (2) $T(0,t) = 0$, $T(L,t) = 0$ for $t > 0$;
  (3) $T(x,0) = f(x)$ for $0 < x < L$,

where the initial temperature profile $f(x)$ is given.

- Solve using Fourier's method:

(I) Use method of separation of variables to find the countably infinite set of nontrivial separable sol^ns satisfying the PDE (1) and BCs (2), each containing an arbitrary constant.

(II) Use the principle of superposition of a linear problem — that the sum of any number of solutions is also a solution (assuming convergence) — to form the general series solution that is the infinite sum of the sep. sol^ns of PDE & BCs.

(III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the IC (3).

- Remarks:

(1) (1) & (2) are linear since, if T_1 and T_2 satisfy them, then so too does $\alpha T_1 + \beta T_2$ $\forall \alpha, \beta \in \mathbb{R}$.

(2) To verify resulting series is actually a solution of PDE, need it to converge suff. rapidly that T_t and T_{xx} can be computed by termwise differentiation — we largely gloss over such issues, i.e. we proceed formally.

(2T)

Step (I)

- $T = F(x)\,G(t) \underset{①}{\Rightarrow} FG' = \kappa F''G \Rightarrow \dfrac{F''}{F} = \dfrac{G'}{\kappa G}$ $\quad (FG \neq 0)$

- LHS ind. t & RHS ind. x $\Rightarrow$ LHS = RHS ind. x & t
  $\Rightarrow$ LHS = RHS $= -\lambda$, say, $\lambda \in \mathbb{R}$.

Hence, $\underline{-F''(x) = \lambda F(x) \text{ for } 0 < x < L}$ $\quad (+)$

- ② $\Rightarrow F(0)\,G(t) = 0$ and $F(L)\,G(t) = 0$ for $t > 0$.

T nontrivial $\Rightarrow G$ nontrivial $\Rightarrow \underline{F(0) = 0,\ F(L) = 0}$ $\quad (\#)$

- Now need to find all $\lambda \in \mathbb{R}$ s.t. ODE BVP $(+)-(\#)$ for $F(x)$ has a nontrivial solution. Consider cases.

(i) $\lambda = -w^2$ $(w > 0$ wlog$)$

$(+) \Rightarrow F'' - w^2 F = 0 \Rightarrow F = A\cosh(wx) + B\sinh(wx)$ $(A, B \in \mathbb{R})$

$(\#) \Rightarrow A = 0,\ B\sinh(wL) = 0 \Rightarrow F = 0.$

(ii) $\lambda = 0$

$(+) \Rightarrow F'' = 0 \Rightarrow F = A + Bx$ $(A, B \in \mathbb{R})$

$(\#) \Rightarrow A = 0,\ BL = 0 \Rightarrow F = 0.$

(iii) $\lambda = w^2$ $(w > 0$ wlog$)$

$(+) \Rightarrow F'' + w^2 F = 0 \Rightarrow F = A\cos(wx) + B\sin(wx)$ $(A, B \in \mathbb{R})$

$(\#) \Rightarrow A = 0,\ B\sin(wL) = 0.$ But $B \neq 0$ for F nontrivial, so $\sin(wL) = 0$, so $wL = n\pi$, $n \in \mathbb{N}\setminus\{0\}$.

(22)

- For $\lambda = \omega^2 = \left(\frac{n\pi}{L}\right)^2$, $F = B\sin\left(\frac{n\pi x}{L}\right)$ and $G \propto \exp\left(-\nu\left(\frac{n\pi}{L}\right)^2 t\right)$
- Hence, nontrivial separable solutions given by

$$T_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\nu t}{L^2}\right),$$

where n is a positive integer and b_n a constant.

Step (II)

- Since ①-② are linear, formally the principal of superposition implies that the general series solution is given by

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\nu t}{L^2}\right).$$

Step (III)

IC ③ can only be satisfied if

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L.$$

The theory of FS ⇒ the Fourier coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad \text{for } n \in \mathbb{N}\setminus\{0\}, \quad (\dagger)$$

which determines the b_n and hence a solution.

Remarks

(1) f, f' p.c. on $(0,L)$ ⇒ sine series converges to $\frac{1}{2}(f(x+) + f(x-))$ for $x \in (0,L)$ and to 0 for $x = 0, L$, so can deal with jump discontinuities in ICs.

(2) In questions often asked to derive (‡‡) via orthogonality relations rather than quoting it. The relevant ones here are

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}\delta_{mn},$$

where $m, n \in \mathbb{N}\setminus\{0\}$. Assuming $\int\Sigma = \Sigma\int$ then gives, for $n \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned}\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx &= \frac{2}{L}\int_0^L \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \sum_{m=1}^{\infty} b_m \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx \\ &= \sum_{m=1}^{\infty} b_m \delta_{mn} \\ &= b_n \end{aligned}$$

□

Example 3.1: $f(x) = \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right)$

$\Rightarrow b_1 = 1,\ b_2 = \frac{1}{2},\ b_n = 0$ otherwise.

Example 3.2: $f(x) = \begin{cases} T^* & \text{for } L_1 < x < L_2 \\ 0 & \text{otherwise} \end{cases}$

$$\Rightarrow b_n = \frac{2}{L}\int_{L_1}^{L_2} T^* \sin\left(\frac{n\pi x}{L}\right)dx = \frac{2T^*}{n\pi}\left(\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right)$$

- We've found a solution (assuming suff. rapid convergence), but is it the only solution?

Uniqueness

Theorem (3.1): The IBVP has only one solution.

Pf: Suppose $T, \tilde{T}$ are solutions and let $W = T - \tilde{T}$.

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By linearity, (1) - (3) $\Rightarrow$

(1') $W_t = T_t - \tilde{T}_t = \kappa T_{xx} - \kappa \tilde{T}_{xx} = \kappa (T - \tilde{T})_{xx} = \kappa W_{xx}$
for $0 < x < L$, $t > 0$;

(2') $W = T - \tilde{T} = 0$ at $x = 0, L$ for $t > 0$;

(3') $W(x,0) = T(x,0) - \tilde{T}(x,0) = f(x) - f(x) = 0$ for $0 < x < L$.

Strategy: deduce that $W(x,t) \equiv 0$.

Trick: analyse $I(t) := \frac{1}{2}\int_0^L W(x,t)^2 \, dx$.

Evidently $I(t) \geq 0$ for $t \geq 0$ and $I(0) = 0$ by (3').

But
$$\frac{dI}{dt} = \int_0^L W W_t \, dx \qquad \text{(by LIR)}$$
$$= \int_0^L W \kappa W_{xx} \, dx \qquad \text{(by (1'))}$$
$$= \left[\kappa W W_x\right]_0^L - \kappa \int_0^L W_x W_x \, dx \qquad \text{(by IBP)}$$
$$= -\kappa \int_0^L W_x^2 \, dx \qquad \text{(by (2'))}$$
$$\leq 0,$$

so $I(t)$ cannot increase!

Hence, $0 \leq I(t) \leq I(0) = 0$, giving $I(t) = 0$ for $t \geq 0$, so that $W = 0$ and $T = \tilde{T}$ for $0 \leq x \leq L$, $t \geq 0$ (assuming cty of W there).

$\square$

- Note that this method of proof works for any linear BCs for which $[W W_x]_0^L \leq 0$, e.g. the radiative BCs $W_x(0,t) = -\alpha W(0,t)$, $W_x(L,t) = \alpha W(L,t)$ for $t > 0$, where α is a positive parameter.

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Non-zero steady state

Example 3.3: Solve the IBVP ① $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;
② $T(0,t) = T_0$, $T(L,t) = T_1$ for $t > 0$;
③ $T(x,0) = 0$ for $0 < x < L$.
where T_0, T_1 are prescribed constants.

- We cannot use separation of variables straight away because BCs are not homogeneous (unless $T_0 = T_1 = 0$).

- Conjecture that $T(x,t) \to S(x)$ as $t \to \infty$, where $S(x)$ is the steady-state solution of ①-②, so that

$$0 = \kappa S_{xx} \text{ for } 0 < x < L \text{ with } S(0) = T_0, \; S(L) = T_1.$$

- Thus, $S(x) = T_0\left(1 - \frac{x}{L}\right) + T_1\left(\frac{x}{L}\right)$, a linear temp. profile.

- Now let $T(x,t) = S(x) + u(x,t)$, then ①-③ $\Rightarrow u(x,t)$ satisfies the IBVP

①' $(S+u)_t = \kappa(S+u)_{xx} \Rightarrow u_t = \kappa u_{xx}$ for $0 < x < L$, $t > 0$;
②' $S(0) + u(0,t) = T_0$, $S(L) + u(L,t) = T_1 \Rightarrow u(0,t) = 0, u(L,t) = 0$ for $t > 0$;
③' $S(x) + u(x,0) = 0 \Rightarrow u(x,0) = -S(x)$ for $0 < x < L$.

- We solved this problem last lecture using Fourier's method:

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right)$$

where $B_n = -\frac{2}{L}\int_0^1 S(x)\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{2}{n\pi}\left(T_0 - (-1)^n T_1\right)$.

□

- Note that T_0, T_1 in BCs ② end up in IC ③' — sometimes called "method of shifting the data."

Other BCs

Example 3.4: Solve the IBVP ① $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;
② $T_x(0,t) = 0$, $T_x(L,t) = 0$ for $t > 0$;
③ $T(x,0) = f(x)$ for $0 < x < L$.

- Note both ends thermally insulated since $q = -kT_x = 0$ at $x = 0, L$.

- Apply Fourier's method on problem sheet 4 $\Rightarrow$

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.

Remarks

(1) The constant separable solution $T = \frac{a_0}{2}$ of ①-② comes from case in which the separation constant is zero.

(2) As $t \to \infty$, $T(x,t) \to \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,dx$, i.e. mean of initial temp.

(3) Uniqueness by similar argument to before.

Inhomogeneous PDE & BCs

Example 3.5: Solve the IBVP ① $\rho c T_t = k T_{xx} + Q(x,t)$ for $0 < x < L$, $t > 0$;
② $T_x(0,t) = \phi(t)$, $T_x(L,t) = \psi(t)$ for $t > 0$;
③ $T(x,0) = f(x)$ for $0 < x < L$;

where $Q(x,t)$, $\phi(t)$, $\psi(t)$ and $f(x)$ are given.

- Note Q is volumetric heat source (e.g. due to radiation or chemical reaction) and heat flux in positive x-direction $q = -kT_x$.

- Now both PDE and BCs are inhomogeneous!

- Deal first with BCs by shifting the data.

- Find $s(x,t)$ s.t. $s_x(0,t) = \phi(t)$, $s_x(L,t) = \psi(t)$ for $t > 0$, e.g. $s(x,t) = -\phi(t)\frac{(x-L)^2}{2L} + \psi(t)\frac{x^2}{2L}$.

- Let $T(x,t) = s(x,t) + u(x,t)$, then ①-③ $\Rightarrow u(x,t)$ satisfies the IBVP

(1') $\rho c u_t = K u_{xx} + \tilde{Q}(x,t)$ for $0 < x < L$, $t > 0$;
(2') $u_x(0,t) = 0$, $u_x(L,t) = 0$ for $t > 0$;
(3') $u(x,0) = \tilde{f}(x)$ for $0 < x < L$;

where $\left.\begin{aligned} \tilde{Q}(x,t) &= Q(x,t) + K s_{xx} - \rho c s_t \\ \tilde{f}(x) &= f(x) - s(x,0) \end{aligned}\right\}$ Known in terms of Q, ϕ, ψ, f.

- If $\tilde{Q} = 0$, then can solve (1')-(3') via Fourier's method as in Example 3.4.

- This suggests we seek a solution of the form

$$u(x,t) = \frac{u_0(t)}{2} + \sum_{n=1}^{\infty} u_n(t) \cos\left(\frac{n\pi x}{L}\right), \qquad (+)$$

where the functions $u_0(t), u_1(t), \ldots$ are TBD.

- Since (+) is a Fourier cosine series, its Fourier coefficients are given by

$$u_n(t) = \frac{2}{L}\int_0^L u(x,t)\cos\left(\frac{n\pi x}{L}\right) dx \quad \text{for } n \in \mathbb{N}.$$

- We can then use (1')-(3') to derive ODEs for the u_n's, as follows.

- By Leibniz's integral rule,

$$\rho c \frac{dU_n}{dt} = \frac{2}{L}\int_0^L \rho c u_t \cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^L (k u_{xx} + \tilde{Q})\cos\left(\frac{n\pi x}{L}\right)dx \qquad \text{(by (1'))}$$

$$= \frac{2k}{L}\int_0^L u_{xx}\cos\left(\frac{n\pi x}{L}\right)dx + \tilde{Q}_n(t),$$

where $\tilde{Q}_n(t) = \frac{2}{L}\int_0^L \tilde{Q}(x,t)\cos\left(\frac{n\pi x}{L}\right)dx$ are the known coefficients of the Fourier cosine series for $\tilde{Q}$.

- How do we deal with the u_{xx} integral? IBP twice via

$$(uv'-u'v)' = uv''-u''v \Rightarrow [uv'-u'v]_a^b = \int_a^b uv''-u''v\,dx.$$

Let $u = u$, $v = \cos\left(\frac{n\pi x}{L}\right)$, $a=0$, $b=L$, then

$$\underbrace{\left[u\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) - u_x\cos\left(\frac{n\pi x}{L}\right)\right]_0^L}_{=0 \text{ by } (2')} = \int_0^L u\left(-\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) - u_{xx}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$\Rightarrow \frac{2}{L}\int_0^L u_{xx}\cos\left(\frac{n\pi x}{L}\right)dx = -\frac{n^2\pi^2}{L^2}\frac{2}{L}\int_0^L u\cos\left(\frac{n\pi x}{L}\right)dx = -\frac{n^2\pi^2}{L^2}U_n.$$

Hence, $\rho c\frac{dU_n}{dt} + \frac{kn^2\pi^2}{L^2}U_n = \tilde{Q}_n(t)$ for $t>0$.

- IC? (3') $\Rightarrow$ $U_n(0) = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

Remarks

(1) Reduced problem to a countably infinite set of ODES — recover solution of Example 3.4 when $Q=0, \phi=0, \psi=0$.

(2) Can solve explicitly for the U_n's using an integrating factor

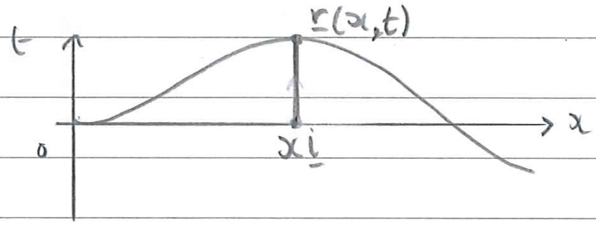
(3) Uniqueness proof the same as for Example 3.4.

The wave equation

Derivation in 1D

- Consider the small transverse vibrations of a homogeneous extensible elastic string stretched initially along the x-axis at time $t = 0$.

- A point at $x\underline{i}$ at time $t = 0$ is displaced to $\underline{r}(x,t) = x\underline{i} + y(x,t)\underline{j}$ at time $t > 0$, where the transverse displacement $y(x,t)$ is TBD.

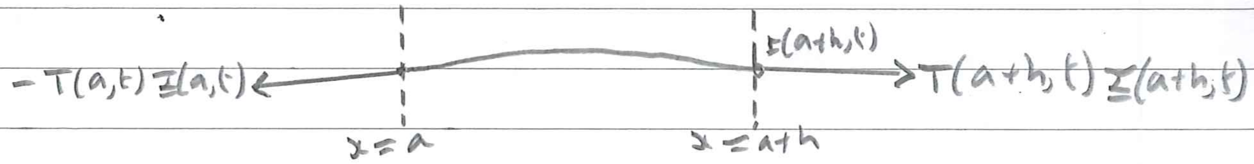


- Consider piece of string in fixed region $a \leq x \leq a+h$.

- Linear momentum is $\int_a^{a+h} \rho \underline{r}_t \, dx$, where ρ is constant line density of the string ($[\rho] = \mathrm{kg\,m^{-1}}$)

- Assuming no resistance to bending (cf a ruler), the string to right of $\underline{r}(x,t)$ exerts at this point a force $T(x,t)\underline{\tau}(x,t)$ on string to left, where $T(x,t)$ is tension ($[T] = N = \mathrm{kg\,m\,s^{-2}}$) and $\underline{\tau} = \underline{r}_x / |\underline{r}_x|$ is unit tangent vector in +ve x-direction.

- Assuming tension so large that gravity and air resistance are negligible, the forces on the string in $a \leq x \leq a+h$ are:



- NII says $\frac{d}{dt}$(linear momentum) = net force, so

$$\frac{d}{dt}\left(\int_a^{a+h} \rho \underline{r}_t \, dx\right) = T(a+h,t)\underline{\tau}(a+h,t) - T(a,t)\underline{\tau}(a,t).$$

- Assuming $\underline{r}_{tt}$ is cts, LIR then gives

$$\frac{1}{h}\int_a^{a+h} \rho \underline{r}_{tt}\, dx = \frac{T(a+h,t)\underline{\tau}(a+h,t) - T(a,t)\underline{\tau}(a,t)}{h}.$$

- Assuming $(T\underline{\tau})_x$ is cts, let $h \to 0$ (from above & below) $\Rightarrow$

$$\rho \underline{r}_{tt} = \frac{\partial}{\partial x}(T\underline{\tau})$$

$$\Rightarrow \quad \rho y_{tt}\underline{j} = \frac{\partial}{\partial x}\left(\frac{T\underline{i} + Ty_x\underline{j}}{(1+y_x^2)^{1/2}}\right)$$

- Now small displacement $\Rightarrow$ small slope $\Rightarrow$ $|y_x| \ll 1$

$$\Rightarrow \quad (1+y_x^2)^{1/2} = 1 + \tfrac{1}{2}(y_x)^2 + \dots$$

$\Rightarrow$ to a first approximation (i.e. neglecting quadratic & h.o.t.)

$$\rho y_{tt} = T_x\underline{i} + (Ty_x)\underline{j}$$

- x-direction $\Rightarrow T_x = 0 \Rightarrow T = T(t)$, i.e. tension is spacially uniform, but could vary with t, e.g. tuning a guitar string. We shall assume T = constant, which is the case in many practical applications.

- y-direction $\Rightarrow \rho y_{tt} = (Ty_x)_x = Ty_{xx}$

- We have derived the <u>wave equation</u>

$$y_{tt} = c^2 y_{xx},$$

where $c = \sqrt{T/\rho}$ is the <u>wave speed</u> (for reasons that will become apparent).