## Units and nondimensionalization

- $[c^2] = \dfrac{[y_{tt}]}{[y_{xx}]} = \dfrac{ms^{-2}}{mm^{-2}} = m^2 s^{-2} \Rightarrow [c] = ms^{-1}.$

Check: $[c^2] = \dfrac{[T]}{[\rho]} = \dfrac{N}{Kgm^{-1}} = \dfrac{Kgms^{-2}}{Kgm^{-1}} = m^2 s^{-2}$ ✓

- On what timescale does a displacement travel a distance $L$?

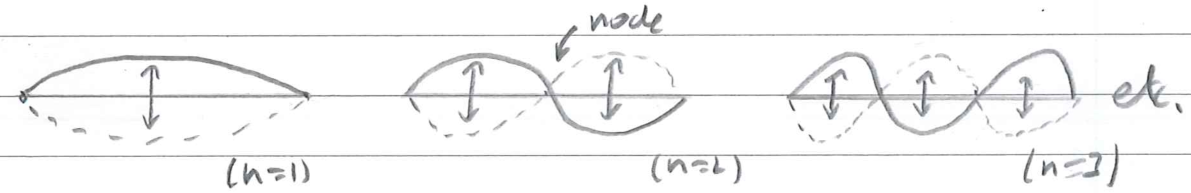
Scale $x = L\hat{x}$, $t = t_0 \hat{t}$, $y = H\hat{y}$

$\Rightarrow \dfrac{H}{t_0^2}\hat{y}_{\hat{t}\hat{t}} = \dfrac{Hc^2}{L^2}\hat{y}_{\hat{x}\hat{x}} \Rightarrow \hat{y}_{\hat{t}\hat{t}} = \hat{y}_{\hat{x}\hat{x}}$ provided $t_0 = \dfrac{L}{c}$.

## Normal modes of vibration for a finite string

- Suppose string stretched between $x=0$ and $x=L$ and the ends held fixed.

- Slinky experiment suggest $\exists$ discrete modes of vibration:




- To analyse mathematically we seek separable solutions to ① $y_{tt} = c^2 y_{xx}$ for $0 < x < L$, $t \in \mathbb{R}$;
  ② $y(0,t) = 0$, $y(L,t) = 0$, $t \in \mathbb{R}$.

- $y = F(x)G(t)$ in ① $\Rightarrow FG'' = c^2 F''G \Rightarrow \dfrac{F''(x)}{F(x)} = \dfrac{G''(t)}{c^2 G(t)}$ $(FG \neq 0)$

- LHS ind. $t$ & RHS ind. $x$ $\Rightarrow$ LHS = RHS ind. $x$ & $t$
  $\Rightarrow$ LHS = RHS $= -\lambda \in \mathbb{R}$, say.

- Hence $-F'' = \lambda F$ for $0 < x < L$ (I)

- ② & $G$ non-trivial $\Rightarrow$ $F(0) = 0$, $F(L) = 0$ (II)

- $\lambda \leq 0 \Rightarrow$ (I) - (II) have only the trivial solution $F=0$.
- Let $\lambda = \omega^2$, with $\omega > 0$ wlog.
- (I) $\Rightarrow$ $F = A\cos(\omega x) + B\sin(\omega x)$ $(A, B \in \mathbb{R})$
- (II) $\Rightarrow$ $A = 0$, $B\sin(\omega L) = 0$
- $F$ nontrivial $\Rightarrow B \neq 0 \Rightarrow \sin(\omega L) = 0 \Rightarrow \omega L = n\pi, n \in \mathbb{N}\setminus\{0\}$.
- $\omega = \frac{n\pi}{L}$ $\Rightarrow$ $F(x) = B\sin\left(\frac{n\pi x}{L}\right)$, $G(t) = C\cos\left(\frac{n\pi c t}{L}\right) + D\sin\left(\frac{n\pi c t}{L}\right)$, $(C, D \in \mathbb{R})$
- Combo $\Rightarrow$ normal modes (nontrivial sep. sol$^{\underline{ns}}$ of ①-②) are

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi c t}{L}\right) + b_n \sin\left(\frac{n\pi c t}{L}\right)\right)$$

or $$y_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c}{L}(t + \varepsilon_n)\right),$$

where $a_n, b_n \in \mathbb{R}$, $c_n, \varepsilon_n \in \mathbb{R}$ for each $n \in \mathbb{N}\setminus\{0\}$.

## Remarks

(1) $y_n$ periodic in $t$ with prime period $p = \frac{2\pi}{n\pi c/L} = \frac{2L}{nc}$ and frequency (or pitch) $\frac{1}{p} = \frac{nc}{2L}$.

(2) $y_1$ is fundamental mode; $\frac{c}{2L}$ the fundamental frequency; all other modes have a frequency that is an integer multiple of $c/2L$.

(3) Consistent with slinky experiment.

(4) Normal modes are an example of a standing wave since $y = f^n(x) \times$ oscillatory function.

[Next time: use Fourier's method to solve IBVP obtained by imposing 2 ICs.]

(33)

## IBVP for a finite string

- Find $y(x,t)$ s.t. ① $y_{tt} = c^2 y_{xx}$ for $0 < x < L, t > 0$;
  ② $y(0,t) = 0$, $y(L,t) = 0$ for $t > 0$;
  ③ $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $0 < x < L$.

- Use Fourier's method. [ $f$ & $g$ = initial transverse displacement & velocity]

Step (I): Find all nontrivial sep. sol$^{\underline{ns}}$ of ①-②

- Last time we found that these (normal modes) are

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right),$$

where $a_n, b_n \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$.

Step (II): Formally apply the principle of superposition

- ① & ② linear so superimpose the normal modes (assuming convergence) to obtain the general series solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t).$$

Step (III): Use theory of FS to satisfy the ICs

- ③ can only be satisfied if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L,$$

$$g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) \quad \text{for } 0 < x < L.$$

- Assuming $\int \sum = \sum \int$, we deduce that

$$\int_0^L f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx}_{=\frac{L}{2}\delta_{nm}} = \frac{L}{2}a_m$$

$$\Rightarrow \quad a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad \text{and similarly}$$

$$b_n\frac{n\pi c}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx. \qquad \square$$

<u>Example 4.1:</u> $f(x) = A\sin\left(\frac{\pi x}{L}\right) + B\sin\left(\frac{2\pi x}{L}\right) \Rightarrow a_1 = A,\ a_2 = B$ and rest 0.

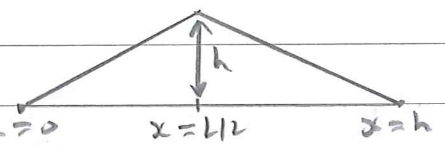
<u>Example 4.2:</u> $f(x) = 0,\ g(x) = \sin^3\left(\frac{\pi x}{L}\right) \Rightarrow a_n = 0\ \forall n.$ $\square$

Trick: $\sin^3\left(\frac{\pi x}{L}\right) = \frac{3}{4}\sin\left(\frac{\pi x}{L}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{L}\right)$

$\Rightarrow \frac{\pi c}{L}b_1 = \frac{3}{4},\ b_2 = 0,\ \frac{3\pi c}{L}b_3 = -\frac{1}{4}$ and rest 0. $\square$

<u>Example 4.3 (Guitar string):</u>

$$f(x) = \begin{cases} 2hx/L & \text{for } 0 \le x \le L/2, \\ 2h(L-x)/L & \text{for } L/2 \le x \le L, \end{cases} \qquad g(x) = 0.$$



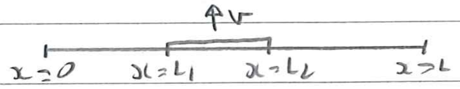

$$a_n = \frac{2}{L}\int_0^{L/2}\frac{2hx}{L}\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^{L}\frac{2h(L-x)}{L}\sin\left(\frac{n\pi x}{L}\right)dx = \frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n = 0$$

Since $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{for } n = 2m,\ m \in \mathbb{N}\setminus\{0\}, \\ (-1)^m & \text{for } n = 2m+1,\ m \in \mathbb{N}, \end{cases}$

we find $y(x,t) = \frac{8h}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi ct}{L}\right)$ $\square$

<u>Example 4.4 (Piano string):</u>




$$f(x) = 0,\quad g(x) = \begin{cases} v & \text{for } L_1 \le L \le L_2, \\ 0 & \text{otherwise.} \end{cases}$$

35

$a_n = 0$ and $b_n = \frac{L}{n\pi c}\frac{2}{L}\int_{L_1}^{L_2} v\sin\left(\frac{n\pi x}{L}\right)$

$$\Rightarrow y(x,t) = \frac{2vL}{c\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\cos\left(\frac{n\pi L_2}{L}\right) - \cos\left(\frac{n\pi L_1}{L_1}\right)\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

$\square$

## Energy and uniqueness

- Consider IBVP ①-③.
- KE of string is $\int_0^L \frac{1}{2}\rho|\underline{r}_t|^2\,dx = \int_0^L \frac{1}{2}\rho y_t^2\,dx$.
- Elastic PE of string is product of tension and extension, i.e.

$$T\left(\int_0^L |\underline{r}_x|\,dx - L\right) = T\int_0^L (1+y_x^2)^{1/2} - 1\,dx.$$

- But $|y_x| \ll 1$, so $(1+y_x^2)^{1/2} - 1 = \frac{1}{2}y_x^2 + \ldots$, so to a first approximation (neglecting cubic and h.o.t.), the elastic PE is $\int_0^L \frac{1}{2}Ty_x^2\,dx$.
- Hence, energy of string $E(t) = \int_0^L \frac{1}{2}\rho y_t^2 + \frac{1}{2}Ty_x^2\,dx$.

Lemma (4.1): If $y$ satisfies ①-②, then $E(t)$ is constant for $t>0$.

Pf: 
$$\frac{dE}{dt} = \int_0^L \rho y_t y_{tt} + T y_x y_{xt}\,dx \qquad \text{(by LIR)}$$
$$= \int_0^L T y_t y_{xx} + T y_x y_{xt}\,dx \qquad \text{(by ①)}$$
$$= \int_0^L (T y_t y_x)_x\,dx$$
$$= \left[T y_t y_x\right]_{x=0}^{x=L}$$
$$= 0,$$

since ② $\Rightarrow$ $y_t(0,t) = y_t(L,t) = 0$ for $t>0$. $\square$

## Remarks

(1) Lemma & ③ $\Rightarrow$ $E(t) = E(0) = \int_0^L \frac{1}{2}\rho(g(x))^2 + \frac{1}{2}T(f'(x))^2\,dx.$

(2) Lemma $\Rightarrow$ Energy in nth normal mode is given by

$$E_n(t) = E_n(0) = \int_0^L \frac{1}{2}\rho(y_{nt}(x,0))^2 + \frac{1}{2}T(y_{nx}(x,0))^2\,dx = \frac{n^2\pi^2 T}{4L}(a_n^2+b_n^2).$$

(3) Can then use Parseval's Identity for $g$ and $f'$ to show that $E(0) = \sum_{n=1}^{\infty} E_n(0)$, i.e. total energy is sum of energy in each normal mode (which are constant throughout motion and set by ICs by remark (2)).

**Theorem (4.2, Uniqueness):** The IBVP has at most one solution.

Pf: Let $W(x,t) = y - \tilde{y}$, where $y, \tilde{y}$ are two solutions, then by linearity,

$$\text{①'}\quad W_{tt} = c^2 W_{xx} \text{ for } 0<x<L,\ t>0;$$
$$\text{②'}\quad W(0,t) = 0,\ W(L,t) = 0 \text{ for } t>0;$$
$$\text{③'}\quad W(0,t) = 0,\ W_t(0,t) = 0 \text{ for } 0<x<L.$$

Lemma (4.1) applied to $W$ $\Rightarrow$

$$\int_0^L \frac{\rho}{2}(W_t)^2 + \frac{T}{2}(W_x)^2\,dx = E(t) \underset{\text{①'②'}}{=} E(0) \underset{\text{③'}}{=} 0 \text{ for } t \geq 0$$

$\Rightarrow$ $W_t = W_x = 0$ for $0<x<L, t>0$ (assuming $W_t, W_x$ cts there)

$\Rightarrow$ $W$ = constant for $0<x<L, t>0$

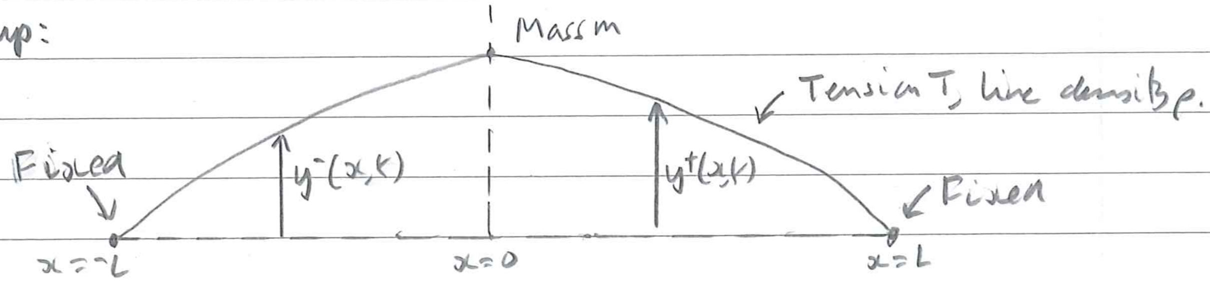
$\Rightarrow$ $W = 0$ for $0 \leq x \leq L, t \geq 0$ (by ②' or ③', assuming $W$ cts for $0 \leq x \leq L, t \geq 0$)

$\square$

37

# Normal modes for a weighted string

- Setup:




- What are the normal modes?

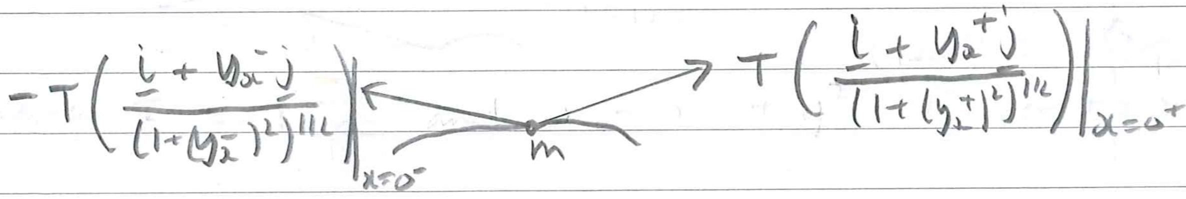
- PDEs: (1⁻) $y^-_{tt} = c^2 y^-_{xx}$ for $-L < x < 0$
  (1⁺) $y^+_{tt} = c^2 y^+_{xx}$ for $0 < x < L$

- BCs: (2⁻) $y^-(-L, t) = 0$
  (2⁺) $y^+(L, t = 0$

  (3) $y^-(0,t) = y^+(0,t) = Y(t)$ say.

- $Y(t)$ TBD so need a second BC at $x = 0$ via NII for the mass.

- Forces on mass (neglecting gravity and air resistance):




- Small transverse displacement $\Rightarrow |y_x^\pm| \ll 1 \Rightarrow (1+(y_x^\pm)^2)^{1/2} = 1 + \text{h.o.t.} \Rightarrow$ to a first approximation mass remains on y-axis (because x-force components balance), while in y-direction

$$④ \quad m\ddot{Y} = T\left(y_x^+|_{x=0^+} - y_x^-|_{x=0^-}\right)$$

- Separate variables: $y^\pm = F_\pm(x)\,G(t)$

38

$(1^{\pm}) \Rightarrow \dfrac{F_{\pm}''(x)}{F_{\pm}(x)} = \dfrac{G''(t)}{c^2 G} = -\lambda \in \mathbb{R}$ say $(F^{\pm}G \neq 0)$

$(2^{\pm})$ & $G \neq 0 \Rightarrow F_-(-L) = 0, \quad F_+(L) = 0$ $(a^{\pm})$

$(3)$ & $G \neq 0 \Rightarrow F_-(0) = F_+(0)$. $(b)$

$(4)$ & $G \neq 0 \Rightarrow m F_{\pm}(0) G''(t) = T(F_+'(0_+) - F_-'(0_-))G(t)$

$\underset{(c^2 = T/\rho)}{\Rightarrow} \quad -\dfrac{\lambda m}{\rho} F_{\pm}(0) = F_+'(0_+) - F_-'(0_-)$ $(c)$

- Can show $\lambda \leq 0 \Rightarrow F_{\pm} = 0$. Let $\lambda = \omega^2$, $\omega > 0$ wlog.
- Then $F_-'' + \omega^2 F_- = 0$ for $-L < x < 0$,
  $F_+'' + \omega^2 F_+ = 0$ for $0 < x < L$.

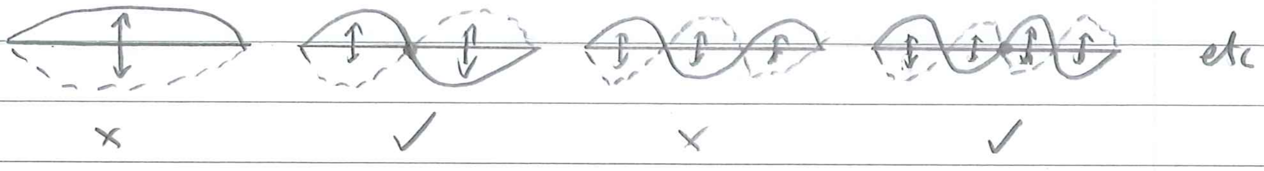
$(a^{\pm}) \Rightarrow F_- = A \sin\omega(L+x), \; F_+ = B\sin\omega(L-x) \quad (A, B \in \mathbb{R})$

$(b)$ & $(c)$ $\Rightarrow \underbrace{\begin{bmatrix} \sin(\omega L) & -\sin(\omega L) \\ \cos(\omega L) - \frac{m\omega}{\rho}\sin(\omega L) & \cos(\omega L) \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $(\dagger)$

- $\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \det(M) = 0 \Rightarrow \sin(\omega L)\left(2\cos(\omega L) - \frac{m\omega}{\rho}\sin(\omega L)\right) = 0$
- Hence, either (i) $\sin(\omega L) = 0 \Rightarrow \omega = \dfrac{n\pi}{L}$, $n \in \mathbb{N}\setminus\{0\}$

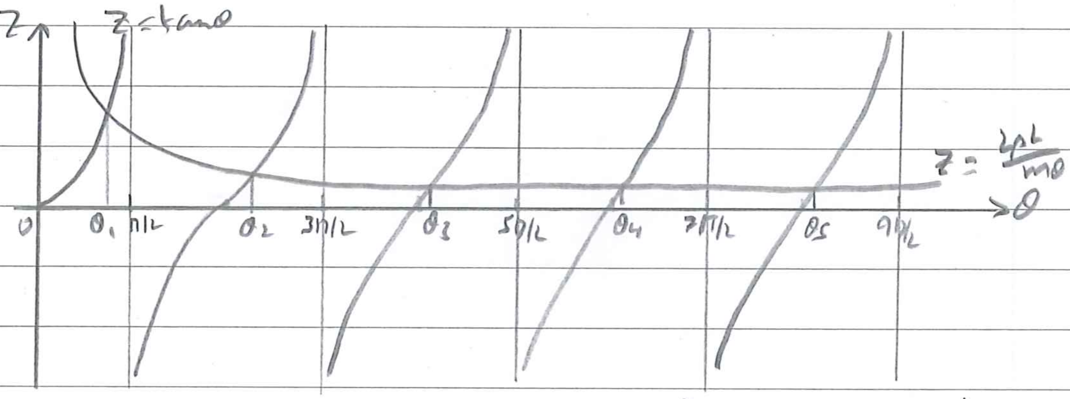
  or (ii) $\cot(\omega L) = \dfrac{m\omega}{2\rho} \Rightarrow \cot\theta = \dfrac{m\theta}{2\rho L}$, where $\theta = \omega L$
- In each case, $G'' + \omega^2 c^2 G = 0 \Rightarrow G(t) = \underset{=1 \text{ wlog}}{C} \cos(\omega c t + \varepsilon) \quad (C, \varepsilon \in \mathbb{R})$
- In case (i), $(\dagger) \Rightarrow A = -B \Rightarrow \begin{cases} y_- = A\sin\omega(L+x)\cos(\omega c t + \varepsilon) \\ y_+ = -A\sin\omega(L-x)\cos(\omega c t + \varepsilon) \end{cases}$

This means that the normal modes are the same as for a string of length $2L$ with a node at $x = 0$, i.e. mass stationary.




- In case (ii) there are infinitely many roots $\theta_1 < \theta_2 < \theta_3 < \dots$

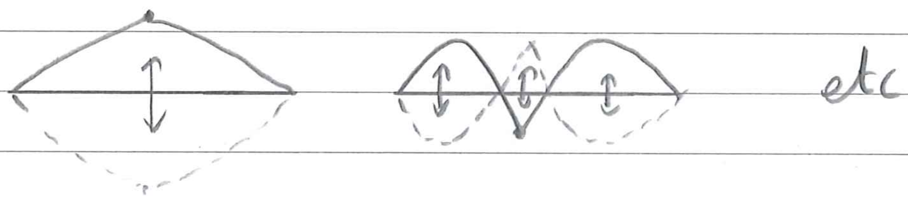
To see plot LHS & RHS of e.g. $\tan\theta = \frac{2\rho L}{m\theta}$




Hence, $\infty$ many normal modes $\omega_n = \frac{\theta_n}{L}$, $n \in \mathbb{N}\setminus\{0\}$.

- Now $(\dagger) \Rightarrow A = B \Rightarrow \begin{cases} y_- = A \sin\omega(L+x)\cos(\omega ct + \varepsilon) \\ y_+ = A \sin\omega(L-x)\cos(\omega ct+\varepsilon) \end{cases}$

This means that the normal modes are symmetric about $x=0$:




- Try with slinky!

## <u>General solution of the wave equation</u>

- Remarkable fact: can write down all solutions of $y_{tt} = c^2 y_{xx}$!

- Let $y(x,t) = Y(\xi,\eta)$, $\xi = x - ct$, $\eta = x + ct$ (as in Intro. Calculus)

$$\Rightarrow \quad y_x = Y_\xi \xi_x + Y_\eta \eta_x = Y_\xi + Y_\eta$$
$$y_{xx} = (Y_\xi + Y_\eta)_\xi \xi_x + (Y_\xi + Y_\eta)_\eta \eta_x = Y_{\xi\xi} + 2Y_{\xi\eta} + Y_{\eta\eta}$$
$$y_t = Y_\xi \xi_t + Y_\eta \eta_t = -cY_\xi + cY_\eta$$
$$y_{tt} = (-cY_\xi + cY_\eta)_\xi \xi_t + (-cY_\xi + cY_\eta)_\eta \eta_t = c^2(Y_{\xi\xi} - 2Y_{\xi\eta} + Y_{\eta\eta})$$

where we assumed $Y_{\xi\eta} = Y_{\eta\xi}$.

Hence, $y_{tt} - c^2 y_{xx} = -4c^2 Y_{\xi\eta}$

Wave equation, $c > 0$ $\Rightarrow Y_{\xi\eta} = 0$

$$\Rightarrow Y_\xi = F'(\xi) \text{ say}$$
$$\Rightarrow (Y - F(\xi))_\xi = 0$$
$$\Rightarrow Y - F(\xi) = G(\eta) \text{ say}$$
$$\Rightarrow y = F(x - ct) + G(x + ct)$$

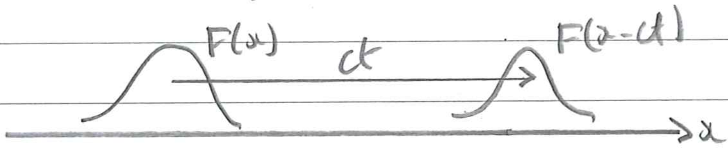
where $F, G$ are arbitrary twice ctsly diff. functions.

<u>Remarks</u>

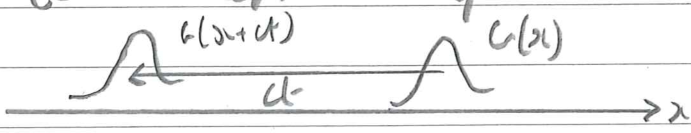
(1) cf. # arb. constants in general solution to a $2^{nd}$ order ODE.

(2) Easy to verify this is a solution (see online notes): we've shown that all solutions must be of this form.

(4) $F(x-ct)$ is a (travelling) wave of constant shape moving to the right with speed $c$:




$G(x+ct)$ is a (travelling) wave of constant shape moving to the left with speed $c$:



41

# Waves on infinite strings: D'Alembert's formula

- Consider the IVP ① $y_{tt} = c^2 y_{xx}$ for $-\infty < x < \infty$, $t > 0$;
  ② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

where initial transverse displacement $f$ and velocity $g$ are given.

- The general solution of ① is $y(x,t) = F(x-ct) + G(x+ct)$.
- ICs ② $\Rightarrow$ $F(x) + G(x) \overset{ⓐ}{=} f(x)$, $-cF'(x) + cG'(x) = g(x)$ for $x \in \mathbb{R}$.

The latter implies $-F(x) + G(x) \overset{ⓑ}{=} \frac{1}{c}\int_0^x g(s)\,ds + a$ $(a \in \mathbb{R})$

ⓐ − ⓑ $\Rightarrow$ $F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds - \frac{1}{2}a$

ⓐ + ⓑ $\Rightarrow$ $G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds + \frac{1}{2}a$

Hence, $y(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2c}\int_{x-ct}^0 g(s)\,ds - \frac{1}{2}a$
$\quad + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \frac{1}{2}a$

$$\Rightarrow \boxed{y(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

D'Alembert's formula

## Remarks

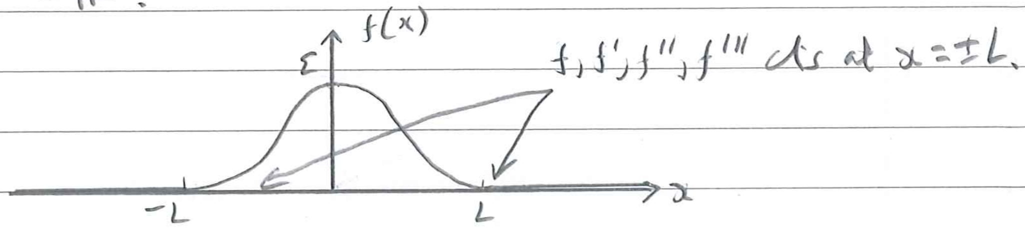
(1) Don't forget the constant $a$!

(2) Argument shows $\exists!$ soln of IVP ①-②.

(3) Can also prove uniqueness via energy conservation under the additional assumption that $y_t, y_x \to 0$ suff. rapidly as $x \to \pm\infty$ that the energy $E(t) = \int_{-\infty}^{\infty} \frac{\rho}{2}y_t^2 + \frac{T}{2}y_x^2\,dx$ exists.

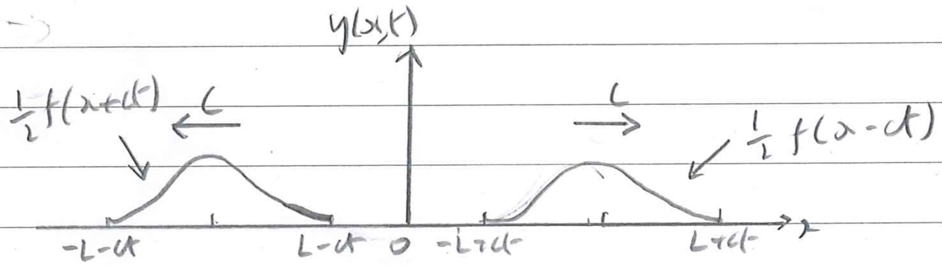
**Example:** $f(x) = \begin{cases} \varepsilon \cos^4\left(\frac{\pi x}{2L}\right) & \text{for } |x| \le L, \\ 0 & \text{otherwise,} \end{cases} \quad g(x) = 0,$

where $\varepsilon, L \in \mathbb{R}^+$.

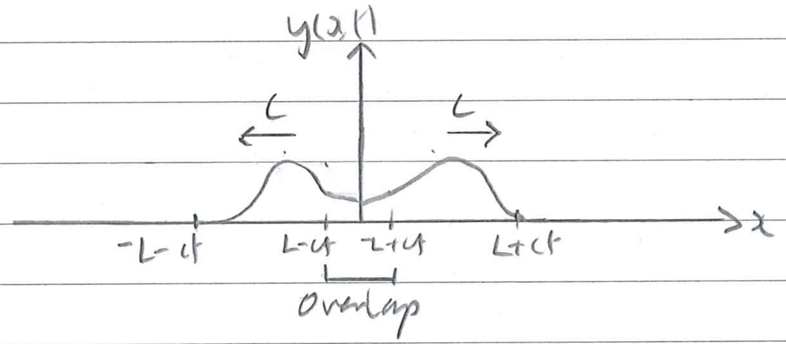



DF $\Rightarrow$ $y(x,t) = \frac{1}{2} f(x-ct) + \frac{1}{2} f(x+ct)$

$ct > L$ $\rightarrow$




$0 < ct < L$




Explicit formulae for these graphs requires some careful book keeping – much easier to use a...

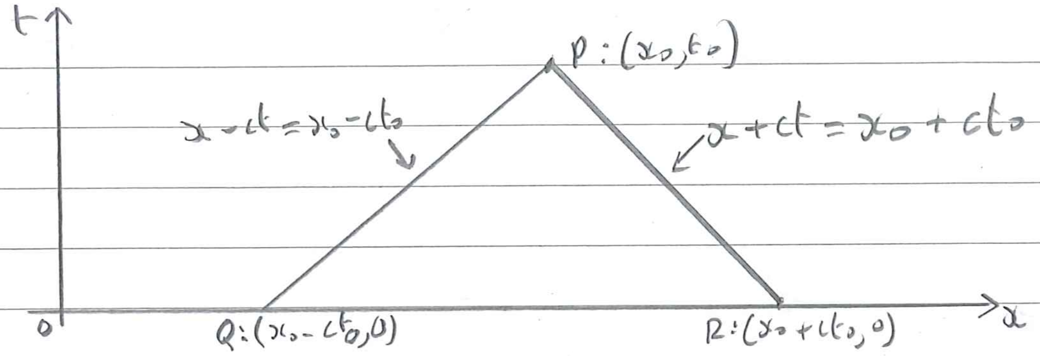
## Characteristic diagram

- Let $P = (x_0, t_0) \in \mathbb{R} \times \mathbb{R}^+$. How does $y(P)$ depend on $f$ & $g$?
- DF $\Rightarrow$ $y(x_0,t_0) = \frac{1}{2}\left(f(x_0-ct_0) + \frac{1}{2} f(x_0+ct_0)\right) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds$ $\quad(+)$

$$\Rightarrow \quad y(P) = \frac{1}{2}\left(f(Q) + f(R)\right) + \frac{1}{2c}\int_Q^R g(s)\,ds, \qquad (++)$$

where $Q$ and $R$ are points on the $x$-axis as shown.

43




- Note deliberate abuse of notation in (H) to aid geometric interpretation of (+).

- Defn: $x \pm ct = x_0 \pm ct_0$ called characteristic lines through P.

- (H) $\Rightarrow$ $y(P)$ depends only on
  (i) $f$ through the values $f$ takes at $Q$ and $R$;
  (ii) $g$ through the values $g$ takes on $x$-axis between $Q$ and $R$.

- Defn: The interval $[x_0 - ct_0, x_0 + ct_0]$ of the $x$-axis between $Q$ and $R$ is called the domain of dependence of $P:(x_0,t_0)$.

- If $f$ or $g$ modified outside the domain of dependence of $P$, then $y(P)$ is unchanged.

- Exploit geometric interpretation (H) of DF (+) to construct explicit formulae for the solution: contribution to $y(P)$ from $f$ and $g$ change at points on $x$-axis where $f$ and $g$ change their analytic behaviour.

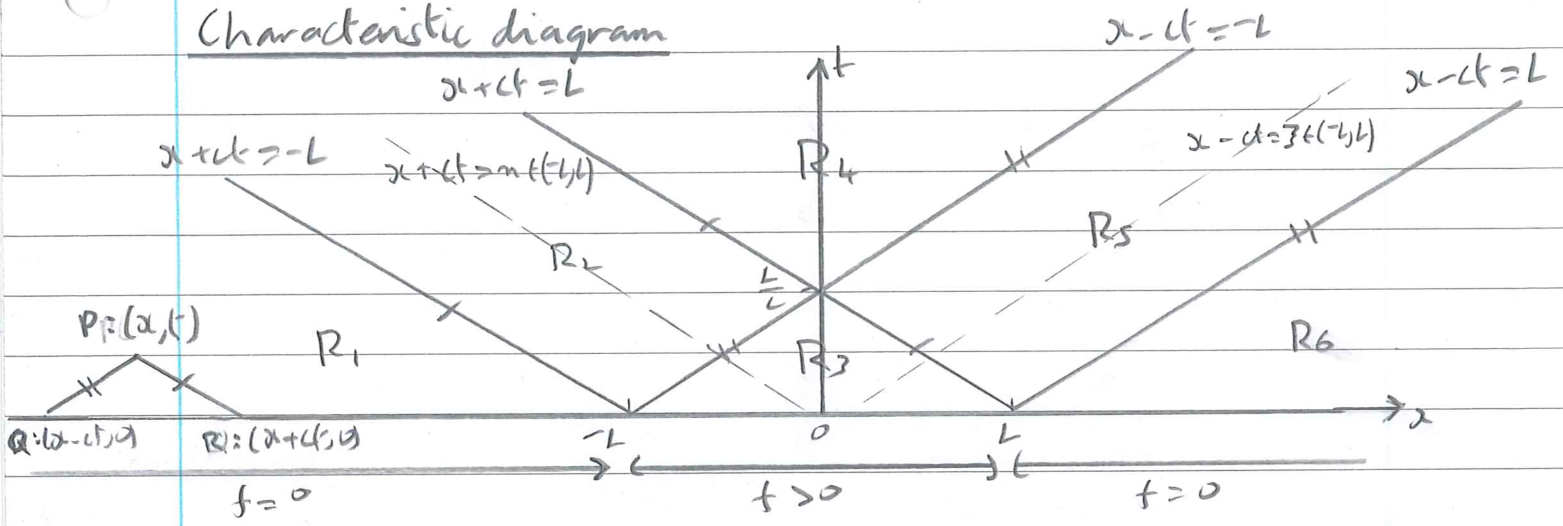
- Hence, given a particular $f$ and $g$, first task is to identify these points on $x$-axis and sketch the characteristic lines $x \pm ct =$ constant through each of them - this is the characteristic diagram.

- This divides the $(x,t)$-plane, with $t > 0$, into regions in which the contributions from $f$ and $g$ may be different.

44

Back to earlier example...

## Characteristic diagram




- DF $\Rightarrow$ $y(P)=\frac{1}{2}\left(f(Q)+f(R)\right)$, where $P,Q,R$ are points shown.
- $PQ \parallel x-ct=\pm L$ and $PR \parallel x+ct=\pm L$, so solution as follows.
- $P\in R_1 \Rightarrow y=\frac{1}{2}[0+0]$
- $P\in R_2 \Rightarrow y=\frac{1}{2}\left[0+\varepsilon\cos^2\left(\frac{\pi}{2L}(x+ct)\right)\right]$
- $P\in R_3 \Rightarrow y=\frac{1}{2}\left[\varepsilon\cos^2\left(\frac{\pi}{2L}(x-ct)\right)+\varepsilon\cos^2\left(\frac{\pi}{2L}(x+ct)\right)\right]$
- $P\in R_4 \Rightarrow y=\frac{1}{2}[0+0]$
- $P\in R_5 \Rightarrow y=\frac{1}{2}\left[\varepsilon\cos^2\left(\frac{\pi}{2L}(x-ct)\right)+0\right]$
- $P\in R_6 \Rightarrow y=\frac{1}{2}[0+0]$
- Since $y$ is continuous on characteristics bounding regions, it does not matter to which region each belongs, e.g. could pick $R_1$: $x+ct<-L$, $t>0$;
  $R_2$: $-L\le x+ct\le L$, $x-ct\le L$;
  $R_3$: $-L<x+ct<L$, $-L<x-ct<L$, $t>0$;

etc.

**Example 4.6**: Suppose $y(x,t)$ s.t. ① $y_{tt} = c^2 y_{xx}$ for $-\infty < x < \infty$, $t > 0$;

② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

Find $y(x,t)$ when $f(x) = 0$ and
$$g(x) = \begin{cases} vx/L & \text{for } |x| \le L, \\ 0 & \text{otherwise,} \end{cases}$$
where $L, v \in \mathbb{R}^+$.

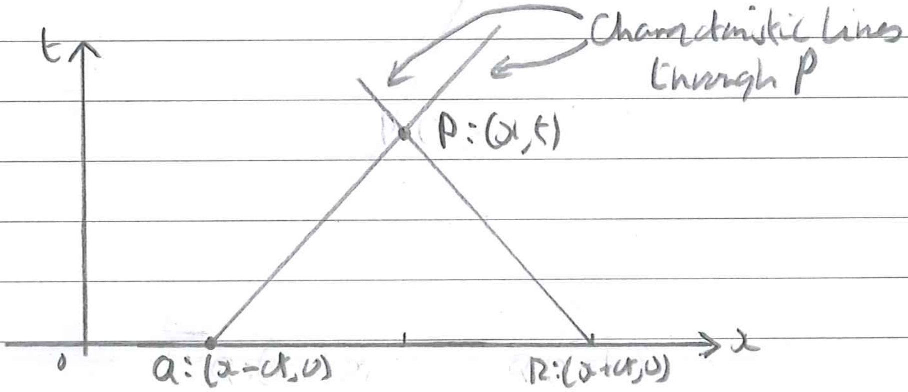
- Recall D'Alembert's Formulae (DF) for the solution of ①-②:

$$y(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

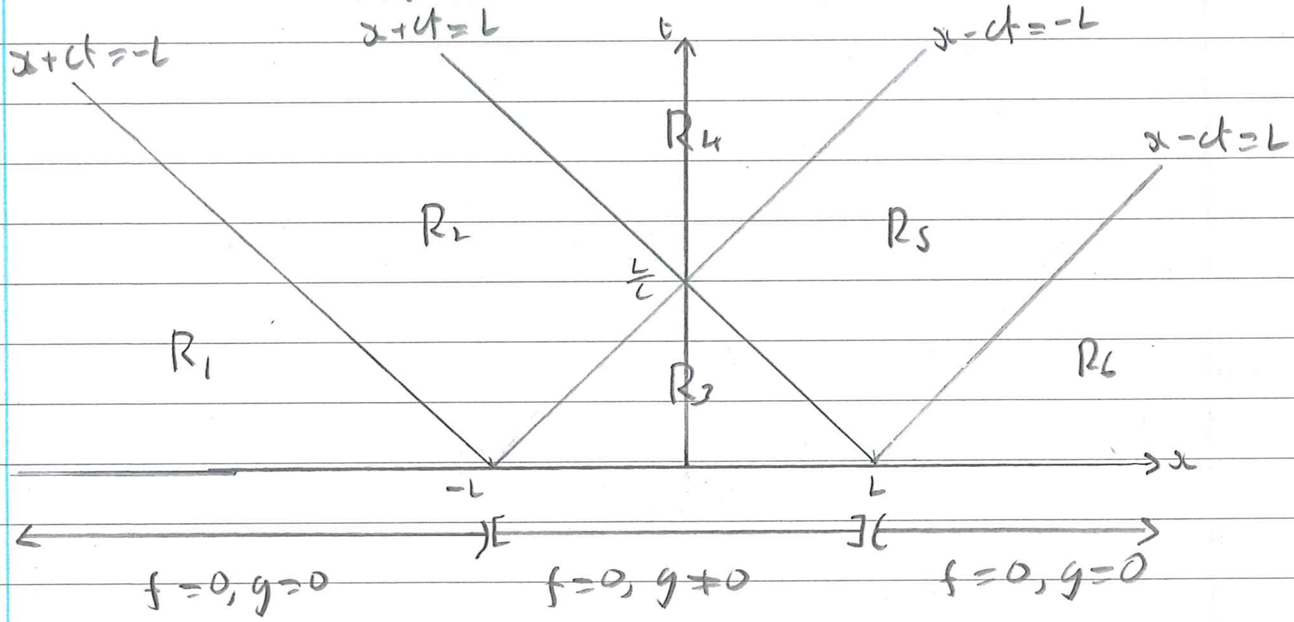
- Thus,

$$y(P) = \frac{1}{2c}\int_{Q}^{R} g(s)\,ds,$$

where P, Q, R are the points shown




**Characteristic diagram**



(46)

- $PQ \parallel x - ct = \pm L$ and $PR \parallel x + ct = \pm L$, so solution as follows.

$$R_1: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} 0 \, ds = 0$$

$$R_2: \quad y = \frac{1}{2c}\int_{x-ct}^{-L} 0 \, ds + \frac{1}{2c}\int_{-L}^{x+ct} \frac{vs}{L} ds = \frac{v}{4Lc}\left((x+ct)^2 - L^2\right)$$

$$R_3: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} \frac{vs}{L} ds = \frac{v}{4Lc}\left((x+ct)^2 - (x-ct)^2\right) = \frac{vxt}{L}$$

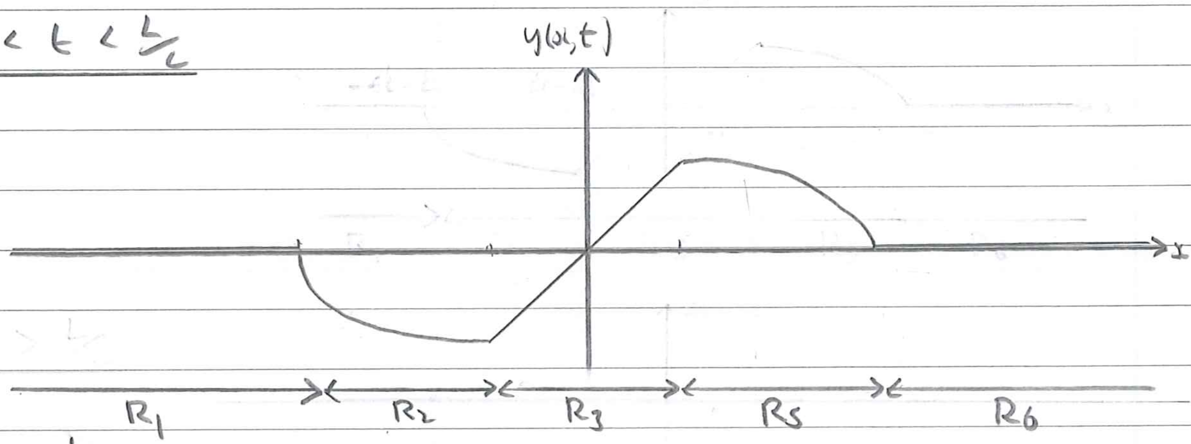
$$R_4: \quad y = \frac{1}{2c}\int_{x-ct}^{-L} 0 \, ds + \frac{1}{2c}\int_{-L}^{L} \frac{vs}{L} ds + \frac{1}{2c}\int_{L}^{x+ct} 0 \, ds = 0$$

$$R_5: \quad y = \frac{1}{2c}\int_{x-ct}^{L} \frac{vs}{L} ds + \frac{1}{2c}\int_{L}^{x+ct} 0 \, ds = \frac{v}{4Lc}\left(L^2 - (x-ct)^2\right)$$

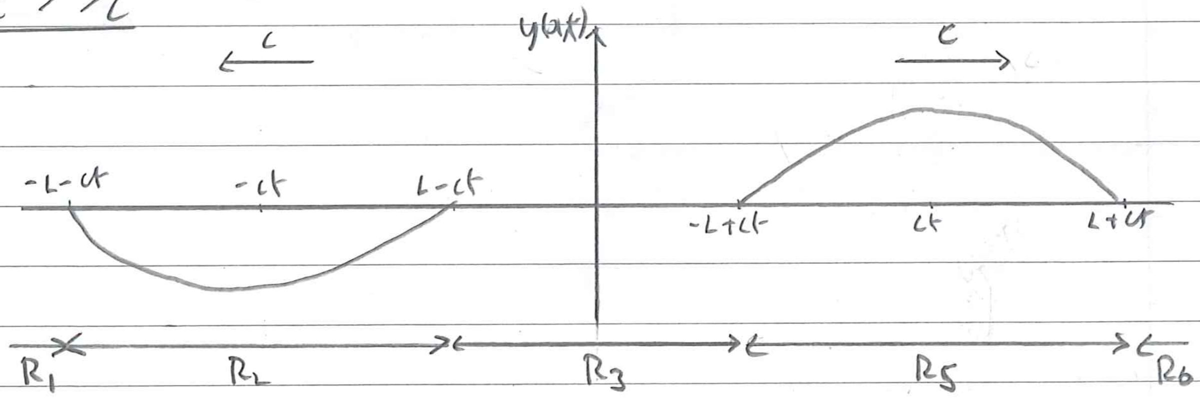
$$R_6: \quad y = \frac{1}{2c}\int_{x-ct}^{x+ct} 0 \, ds = 0$$

- Note solution cts across borders between regions.

$0 < t < L/c$




$t > L/c$




- Note ∃ corners ⇒ not a classical (twice cts diff.) solution!