(49)

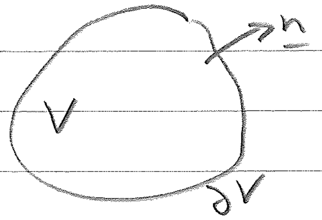
Laplace's equation in the plane

- Heat conduction in a rigid isotropic material (e.g. metal) is governed in 3D by the heat equation

$$T_t = \kappa \nabla^2 T,$$

where $T(x,y,z,t)$ is the temperature, κ the thermal diffusivity and $\nabla^2 T = T_{xx} + T_{yy} + T_{zz}$.

- Derive in Multivariable Calculus from conservation of energy and Fourier's Law using the Divergence Theorem:



$$\frac{d}{dt}\iiint_V \rho c T \, dV = \iint_{\partial V} \underline{q}\cdot(-\underline{n})\, dS \qquad \text{[Energy]}$$

$$\underset{\text{(Div. Thm)}}{\Rightarrow} \iiint_V \rho c T_t \, dV = -\iiint_V \nabla\cdot\underline{q}\, dV$$

$$\Rightarrow \rho c T_t + \nabla\cdot\underline{q} = 0 \text{ (assuming LHS cts)}$$

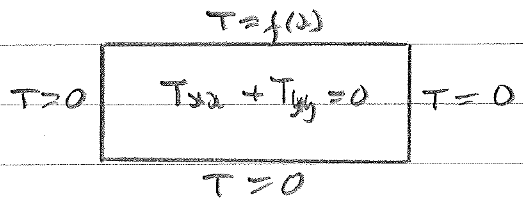
Substitute $\underset{\text{(Fourier's Law)}}{\underline{q} = -k\nabla T} \Rightarrow T_t = \frac{k}{\rho c}\nabla\cdot\nabla T = \kappa \nabla^2 T$ □

- In this course we consider 2D steady-state solutions:

$$T = T(x,y) \Rightarrow \boxed{\begin{array}{c} T_{xx} + T_{yy} = 0 \\ \text{Laplace's equation} \end{array}}$$

BVP in Cartesian Coordinates

Find $T(x,y)$ s.t. ① $T_{xx} + T_{yy} = 0$ for $0 < x < a$, $0 < y < b$;
② $T(0,y) = 0$, $T(a,y) = 0$ for $0 < y < b$;
③ $T(x,0) = 0$, $T(x,b) = f(x)$ for $0 < x < a$.



Apply Fourier's method.

Step (I): $T = F(x)G(y)$ in ① $\Rightarrow \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}$ $\quad (FG \neq 0)$

LHS ind. y & RHS ind. $x \Rightarrow$ LHS = RHS ind. x & $y \Rightarrow$ LHS = RHS = $-\lambda \in \mathbb{R}$, say.

Hence $-F'' = \lambda F$ for $0 < x < a$. ② & T nontrivial $\Rightarrow F(0) = 0$, $F(a) = 0$.

Solved before! Only nontrivial solutions are $F(x) = B\sin\left(\frac{n\pi x}{a}\right)$ ($B \in \mathbb{R}$) for $\lambda = \left(\frac{n\pi}{a}\right)^2$, $n \in \mathbb{N} \setminus \{0\}$.

$\lambda = \frac{n\pi}{a} \Rightarrow G'' - \left(\frac{n\pi}{a}\right)^2 G = 0 \Rightarrow G = C\cosh\left(\frac{n\pi y}{a}\right) + D\sinh\left(\frac{n\pi y}{a}\right)$ $(C, D \in \mathbb{R})$

Combo $\Rightarrow$ nontrivial sep. solns of ①-② are

$$T_n(x,y) = \sin\left(\frac{n\pi x}{a}\right)\left(a_n\cosh\left(\frac{n\pi y}{a}\right) + b_n\sinh\left(\frac{n\pi y}{a}\right)\right),$$

where $a_n, b_n \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$.

Step (II): Formally superimpose $\Rightarrow T(x,y) = \sum_{n=1}^{\infty} T_n(x,y)$ is the general series solution of ①-②

Step (III): BC on $y = 0 \Rightarrow 0 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \Rightarrow a_n = 0 \ \forall n$.

BC on $y = b \Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$ for $0 < x < a$,

so that $b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$ by theory of FS.

NB: Could also apply BC on $y = 0$ to find $a_n = 0$ at end of step (I), i.e. before superimposing in step (II).

NB: On sheet consider case in which $a = b = L$ and $f = T^* \in \mathbb{R}$

(51)

BVP in plane polar coordinates

- In plane polar coordinates (r, θ), Laplace's equation for $T(r,\theta)$ becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{for } r > 0. \quad (*)$$

- Start by finding all nontrivial separable solutions that are 2π-periodic in θ.

- $T = F(r)G(\theta) \Rightarrow F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0$

$$\underset{(\times \frac{r^2}{FG})}{\Rightarrow} \frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} \qquad (FG \neq 0)$$

- LHS ind. θ & RHS ind. $r \Rightarrow$ LHS = RHS ind. r & $\theta \Rightarrow$ LHS = RHS = $\lambda \in \mathbb{R}$.

- Hence, need to find all $\lambda \in \mathbb{R}$ s.t. $G''(\theta) + \lambda G(\theta) = 0$ has a nontrivial solution $G(\theta)$ of period 2π. Consider cases.

(i) $\lambda = -\omega^2$ ($\omega > 0$ wlog) $\Rightarrow G(\theta) = A\cosh(\omega\theta) + B\sinh(\omega\theta)$ $(A, B \in \mathbb{R})$

- G 2π-periodic $\Rightarrow G(0) = G(\pm 2\pi) \Rightarrow A = A\cosh(2\pi\omega) \pm B\sinh(2\pi\omega)$
$\underset{(+,-)}{\Rightarrow} A(\cosh(2\pi\omega) - 1) = 0,\ B\sinh(2\pi\omega) = 0 \underset{(\omega>0)}{\Rightarrow} A = B = 0 \Rightarrow G = 0.$

(ii) $\lambda = 0 \Rightarrow G(\theta) = A + B\theta$ $(A, B \in \mathbb{R})$.

- G 2π-periodic $\Rightarrow B = 0$, but A arbitrary admissible.

- $r^2F'' + rF' = 0 \Rightarrow r(rF')' = 0$
$$\Rightarrow rF' = d \qquad (r > 0, d \in \mathbb{R})$$
$$\Rightarrow F = c + d\log r \qquad (c \in \mathbb{R})$$

- Combo $\Rightarrow T_0 = A_0 + B_0 \log r \quad (A_0, B_0 \in \mathbb{R})$
This is a cylindrically symmetric solution (i.e. ind. of θ).

(iii) $\lambda = \omega^2$ ($\omega > 0$ wlog) $\Rightarrow G(\theta) = R\cos(\omega\theta + \Phi)$ $(R, \Phi \in \mathbb{R})$

- G nontrivial $\Rightarrow R \neq 0 \Rightarrow G$ has prime period $\frac{2\pi}{\omega}$.

Hence, G 2π-periodic and non-trivial $\Rightarrow \exists n \in \mathbb{N}\setminus\{0\}$ s.t. $n \cdot \frac{2\pi}{\omega} = 2\pi$, i.e. $\omega = n$ for some $n \in \mathbb{N}\setminus\{0\}$.

In practice, better to write $G(\theta) = A\cos(n\theta) + B\sin(n\theta)$, where $A = R\cos\Phi$, $B = -R\sin\Phi$ are arb. real constants.

- $\lambda = n^2 \Rightarrow r^2F'' + rF' - n^2F = 0$ for $r > 0$. (Euler's ODE)

Let $r = e^t$, $F(r) = W(t)$, then $\frac{dW}{dt} = \frac{dF}{dr}\frac{dr}{dt} = r\frac{dF}{dr}$, so

$$\frac{d^2W}{dt^2} = \frac{d}{dr}\left(r\frac{dF}{dr}\right)\frac{dr}{dt} = r\frac{d}{dr}\left(r\frac{dF}{dr}\right) = r^2F'' + rF' = n^2F = n^2W$$

$W = e^{\mu t} \Rightarrow \mu^2 = n^2 \Rightarrow W$ has general solution $W = Ce^{nt} + De^{-nt}$ $(C, D \in \mathbb{R}) \Rightarrow F$ has general solution

$$F(r) = Cr^n + Dr^{-n} \quad (C, D \in \mathbb{R}).$$

NB: Alternatively, let $F(r) = r^\mu$, then $\mu(\mu-1) + \mu - n^2 = 0$ $\Rightarrow \mu^2 = n^2 \Rightarrow \mu = \pm n \Rightarrow$ general solution as above by theory of linear 2nd order ODEs.

- Combo $\Rightarrow T_n = (A_nr^n + B_nr^{-n})\cos(n\theta) + (C_nr^n + D_nr^{-n})\sin(n\theta)$,

where $A_n = AC$, $B_n = AD$, $C_n = BC$, $D_n = BD$ are arb. real constants and $n \in \mathbb{N}\setminus\{0\}$.

- Superimpose $\Rightarrow$ general series solution of $(*)$ is

$$T(r,\theta) = \sum_{n=0}^{\infty} T_n(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\Big((A_nr^n + B_nr^{-n})\cos(n\theta) + (C_nr^n + D_nr^{-n})\sin(n\theta)\Big).$$

BVP in plane polar coordinates ctd

- Last lecture we showed that the general series solution of

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \qquad (r>0)$$

is given by

$$T(r,\theta) = A_0 + B_0 \log r + \sum_{n=1}^{\infty}\left((A_n r^n + B_n r^{-n})\cos(n\theta) + (C_n r^n + D_n r^{-n})\sin(n\theta)\right) \qquad (*)$$

where $A_n, B_n, C_n, D_n \in \mathbb{R}$.

<u>Example 5.1</u>: Find T s.t. ① $\nabla^2 T = 0$ in $a<r<b$,
② $T = T_0^*$ on $r=a$, $T = T_1^*$ on $r=b$

where $a, b, T_0^*, T_1^* \in \mathbb{R}$.

- ① $\Rightarrow$ $(*)$ pertains. BCs ② can only be satisfied if

$$T_0^* = A_0 + B_0 \log a + \sum_{n=1}^{\infty}\left((A_n a^n + B_n a^{-n})\cos(n\theta) + (C_n a^n + D_n a^{-n})\sin(n\theta)\right)$$

$$T_1^* = A_0 + B_0 \log b + \sum_{n=1}^{\infty}\left((A_n b^n + B_n b^{-n})\cos(n\theta) + (C_n b^n + D_n b^{-n})\sin(n\theta)\right)$$

each for $-\pi < \theta \le \pi$, say.

- Since the Fourier coefficients of a Fourier series are unique, we can equate them on LHS & RHS of each equality $\Rightarrow$

$$\begin{bmatrix} 1 & \log a \\ 1 & \log b \end{bmatrix}\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} T_0^* \\ T_1^* \end{bmatrix}, \quad \underbrace{\begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix}\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix}\begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{n \in \mathbb{N}\setminus\{0\}}$$

$$\underset{(a<b)}{\Rightarrow} \quad \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \frac{1}{\log(b/a)}\begin{bmatrix} \log b & -\log a \\ -1 & 1 \end{bmatrix}\begin{bmatrix} T_0^* \\ T_1^* \end{bmatrix}, \quad \underbrace{\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},}_{n \in \mathbb{N}\setminus\{0\}}$$

$$\Rightarrow \; T = \frac{T_0^* \log b - T_1^* \log a}{\log(b/a)} + \frac{T_1^* - T_0^*}{\log(b/a)} \log r$$

Dimensionally correct !?

$$T = T_0^* \frac{\log\left(\frac{r}{b}\right)}{\log\left(\frac{a}{b}\right)} + T_1^* \frac{\log\left(\frac{r}{a}\right)}{\log\left(\frac{b}{a}\right)}$$

$\square$

NB: Alternatively, we could have sought a circularly symmetric solution $T = T(r)$ from the outset because boundary data is ind. of θ. However, method above generalizes to T_0^* and T_1^* being functions of θ.

<u>Example 5.2</u> : Find T s.t. ① $\nabla^2 T = 0$ in $r < a$,
② $T(a, \theta) = T^* \sin^3\theta$ for $-\pi < \theta \leq \pi$,
where $a, T^* \in \mathbb{R}^+$.

- ① $\Rightarrow$ T must be twice differentiable wrt x and y at origin

  $\Rightarrow$ T must certainly be cts and therefore <u>bounded</u> at origin

  $\Rightarrow$ (*) pertains but with $B_n = 0 \; \forall n \in \mathbb{N}$ and $D_n = 0 \; \forall n \in \mathbb{N}\cup\{0\}$

- ② then requires.

$$T^* \sin^3\theta = A_n + \sum_{n=1}^{\infty} \left(A_n a^n \cos(n\theta) + B_n a^n \sin(n\theta) \right) \text{ for } -\pi < \theta \leq \pi$$

But the FS for the LHS is given by the identity

$$T^* \sin^3\theta = \frac{3T^*}{4} \sin\theta - \frac{T^*}{4} \sin(3\theta),$$

so equating Fourier coefficients gives

$$B_1 a = \frac{3T^*}{4}, \quad B_3 a^3 = -\frac{T^*}{4} \text{ and rest vanish.}$$

Hence, $T = \frac{3T^*}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{T^*}{4}\left(\frac{r}{a}\right)^3 \sin 3\theta$

Q$^{\underline{n}}$: What is the heat flux out of the disc through $r=a$?

A$^{\underline{ns}}$: Outward pointing unit normal $\underline{n} = \underline{e}_r$, so by Fourier's Law

$$\underline{q}\cdot\underline{n}\big|_{r=a} = (-k\nabla T)\cdot\underline{e}_r\big|_{r=a} = -kT_r(a,\theta) = -k\left(\frac{3T^*}{4a}\sin\theta - \frac{3T^*}{4a}\sin 3\theta\right)$$

NB: $\nabla^2 T = 0 \Leftrightarrow \nabla\cdot\underline{q} = 0 \Rightarrow \int_{r=a} \underline{q}\cdot\underline{n}\,ds = \iint_{r<a} \nabla\cdot\underline{q}\,dx\,dy = 0$,

so zero net heat flux through $r=a$ as there's no volumetric heating

Poisson's Integral Formula

- Find T s.t. $\nabla^2 T = 0$ in $r<a$ with $T(a,\theta) = f(\theta)$ for $-\pi<\theta\leq\pi$, where $a\in\mathbb{R}^+$ and f is given.

- As in last example, general series solution is given by $(*)$ with $B_n = 0\ \forall n\in\mathbb{N}$, $D_n = 0\ \forall n\in\mathbb{N}\setminus\{0\}$, so BC satisfied if

$$f(\theta) = \underbrace{A_0}_{a_0/2} + \sum_{n=1}^{\infty}\left(\underbrace{A_n a^n}_{a_n}\cos(n\theta) + \underbrace{B_n a^n}_{b_n}\sin(n\theta)\right) \text{ for } -\pi<\theta\leq\pi.$$

- Theory of FS then gives

$$A_0 = \frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$$

$$A_n = \frac{a_n}{a^n} = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,d\phi \qquad (n\in\mathbb{N}\setminus\{0\})$$

$$B_n = \frac{b_n}{a^n} = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,d\phi \qquad (n\in\mathbb{N}\setminus\{0\})$$

where we introduced a dummy variable ϕ for convenience below.

- Given a particular f, can evaluate these expressions (see e.g. Example 5.2 in online notes), but remarkably we can sum for general f (suff. smooth that following OIT).

- Sub. Fourier coeffs into general series soln and assume $\sum \int = \int \sum$ gives

$$T(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^n\left(\cos(n\theta)\cos(n\phi)+\sin(n\theta)\sin(n\phi)\right)f(\phi)\,d\phi\right)$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta-\phi)\right) f(\phi)\,d\phi$$

- Let $\alpha = \theta - \phi$ and $z = \frac{r}{a}e^{i\alpha}$, then

$$\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos(n\alpha) = \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{in\alpha}\right)$$

$$= \mathrm{Re}\left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right)$$

$$= \mathrm{Re}\left(\frac{1}{2} + \frac{z}{1-z}\right) \qquad (|z|<1)$$

$$= \frac{1}{2}\,\mathrm{Re}\left(\frac{1+z}{1-z}\right)$$

$$= \frac{a^2 - r^2}{2(a^2 - 2ar\cos\alpha + r^2)} \qquad \left(z = \frac{a}{r}e^{i\alpha}\right)$$

- Hence, obtain PIF:

$$T(r,\theta) = \frac{(a^2-r^2)}{2\pi}\int_{-\pi}^{\pi}\frac{f(\phi)\,d\phi}{a^2 - 2ar\cos(\theta-\phi) + r^2} \qquad (r<1)$$

- NB: $r = 0 \Rightarrow T = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,d\phi$

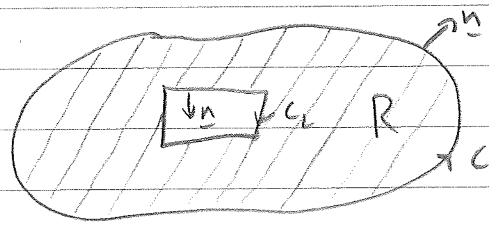
This means temp at centre of disc is average of the temp. profile on the boundary.

- Next time: uniqueness.

Uniqueness

(Handout) **Green's Theorem in the plane (Divergence Theorem Form)**

Let R be a closed bounded region in the (x,y)-plane, whose boundary ∂R is the union $C_1 \cup C_2 \cup \ldots \cup C_m$ of a finite number of piecewise smooth simple closed curves.



Let $\underline{F} = (F_1(x,y), F_2(x,y))$ be continuous and have continuous first order derivatives on $R \cup \partial R$. Then

$$\iint_R \nabla \cdot \underline{F} \, dx\,dy = \int_{\partial R} \underline{F} \cdot \underline{n} \, ds,$$

where $\underline{n}$ is the outward pointing unit normal to ∂R in the (x,y)-plane and ds an element of arclength.

Example: Derivation of the 2D inhomogeneous heat equation

$$[\text{Energy}]: \quad \underbrace{\frac{d}{dt}\iint_R \rho c T \, dx\,dy}_{\substack{\text{Rate of change of}\\ \text{internal heat energy}}} = \underbrace{\int_{\partial R} \underline{q} \cdot (-\underline{n})\, ds}_{\substack{\text{Net heat flux}\\ \text{into } R \text{ through } \partial R}} + \underbrace{\iint_R Q \, dx\,dy}_{\substack{\text{Volumetric}\\ \text{heating}}}$$

NB: [Each term] $= \mathrm{J\,m^{-1}\,s^{-1}}$ since this is per unit distance in z-direction.

Assuming T_t continuous on $R \cup \partial R$ & using Green's Thm with $\underline{F} = \underline{q}$ gives
$$\iint_R \rho c T_t + \nabla \cdot \underline{q} - Q \, dx\,dy = 0$$

Assuming integrand continuous, R arbitrary $\Rightarrow \rho c T_t + \nabla \cdot \underline{q} = Q$

Finally, Fourier's law $\underline{q} = -k \nabla T$ gives $\rho c T_t = \nabla \cdot (k \nabla T) + Q$ ◻

Uniqueness for the Dirichlet problem

Thm: Suppose $T(x,y)$ s.t. $\nabla^2 T = 0$ in R with $T = f$ on ∂R (Dirichlet problem), where R as in Green's Thm and path-connected and f given. Then the BVP has at most one solution.

Pf: Let W be the difference between two solutions, then linearity gives ① $\nabla^2 W = 0$ in R;
② $W = 0$ on ∂R.

Trick: let $\underline{F} = W\nabla W = \nabla(\frac{1}{2}W^2)$ in Green's Thm.

Then $$\iint_R \nabla\cdot(W\nabla W)\,dx\,dy = \int_{\partial R} W\nabla W\cdot\underline{n}\,ds$$

But ① $\Rightarrow$ $\nabla\cdot(W\nabla W) = W\nabla^2 W + \nabla W\cdot\nabla W = |\nabla W|^2$ in R

② $\Rightarrow$ $W\nabla W\cdot\underline{n} = 0$ on ∂R

so $$\iint_R |\nabla W|^2\,dx\,dy = 0.$$

Assuming ∇W is continuous on $R\cup\partial R$, this implies $\nabla W = \underline{0}$ on $R \Rightarrow W =$ constant on R (as it's path-connected).

But $W = 0$ on ∂R, so assuming W is continuous on $R\cup\partial R$, the constant must vanish, so that $W = 0$ on $R\cup\partial R$ □

Example: Find T s.t. $\nabla^2 T = 0$ in $r < a$ with $T = T^* \frac{x}{a}$ on $r = a$

If we can find any solution, then uniqueness thm guarantees it is the only solution.

Could proceed via general series solution or Poisson's Integral Formula, but quicker to spot $T = T^*\frac{x}{a}$.

<u>Example</u>: Find T s.t. $\nabla^2 T = 0$ in $r > a$ with $T = T^* \frac{x}{a}$ on $r = a$ and T bounded as $r \to \infty$.

Spot $T = B_1 r^{-1} \cos\theta$ is a solution provided $B_1 a^{-1} = T^*$.

<u>Qn</u>: Is it the only solution?

<u>Ans</u>: Uniqueness thm above not applicable because R not bdd.

But if W is difference between two solutions, then for fixed $b > a$

$$\iint_{a<r<b} |\nabla W|^2 \,dx\,dy \underset{\substack{\uparrow \\ \nabla^2 W = 0 \text{ in} \\ a<r<b}}{=} \iint_{a<r<b} \nabla\cdot(W\nabla W)\,dx\,dy = \underbrace{\int_{r=b} W\nabla W.\underline{e}_r\,ds}_{W\frac{\partial W}{\partial r}b\,d\theta} - \underbrace{\int_{r=a} W\nabla W.\underline{e}_r\,ds}_{=0 \text{ since } W=0 \text{ at } r=a}$$

So have uniqueness provided $rW\frac{\partial W}{\partial r} \to 0$ as $r \to \infty$, which is the case if e.g. $r\frac{\partial T}{\partial r} \to 0$ as $r \to \infty$. □

<u>Uniqueness for the Neumann Problem</u>

<u>Thm</u>: Suppose $T(x,y)$ s.t. $\nabla^2 T = F$ in R with $\frac{\partial T}{\partial n} \equiv \underline{n}\cdot\nabla T = g$ on ∂R (Neumann problem), where R as in Green's Thm and path-connected and F, g given. Then the BVP has no solution unless

$$\iint_R F\,dx\,dy = \int_{\partial R} g\,ds.$$

When a solution exists, it is not unique: any two solutions differ by a constant.

<u>Pf</u>: Suppose there is a solution T and let $\underline{F} = \nabla T$ in Green's Thm, then

$$\iint_R F\,dx\,dy = \iint_R \nabla\cdot(\nabla T)\,dx\,dy = \int_{\partial R} \nabla T\cdot\underline{n}\,ds = \int_{\partial R} g\,ds.$$

Now let W be the difference between two solutions, so that $\nabla^2 W = 0$ in R and $\frac{\partial W}{\partial n} = 0$ in R. Then

$$\iint_R |\nabla W|^2 \, dx dy \underset{\uparrow \text{PDE}}{=} \iint_R \nabla \cdot (W \nabla W) \, dx dy \underset{\uparrow \text{Green's Thm}}{=} \int_{\partial R} W \nabla W \cdot \underline{n} \, ds \underset{\uparrow \text{BC}}{=} 0 \text{ as before.}$$

Assuming ∇W is cts on $R \cup \partial R$, this implies $\nabla W = \underline{0}$ on R, so that $W =$ constant on R. Hence, $W =$ constant on $R \cup \partial R$ assuming W cts there. □

<u>Example</u>: Find T s.t. $\nabla^2 T = 0$ in $r < a$ with $\frac{\partial T}{\partial n} = g(\theta)$ on $r = a$, where g is given.

General series solution of $\nabla^2 T = 0$ in $r < a$ is

$$T = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \right).$$

On $r = a$, $\frac{\partial T}{\partial n} = \underline{n} \cdot \nabla T = \frac{\partial T}{\partial r}$, so BC can satisfied only if

$$g(\theta) = \sum_{n=1}^{\infty} \left(n A_n a^{n-1} \cos(n\theta) + n B_n a^{n-1} \sin(n\theta) \right) \text{ for } -\pi < \theta \leq \pi \text{ (say)}$$

Theory of FS then requires

$$0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \, d\theta \qquad (+)$$

$$n A_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta) \, d\theta \qquad (n \in \mathbb{N} \setminus \{0\})$$

$$n B_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin(n\theta) \, d\theta \qquad (n \in \mathbb{N} \setminus \{0\})$$

Two cases.

Either (i) g s.t. $(+)$ not true, in which case there is no solution;

or (ii) g s.t. $(+)$ is true, in which case the solution is not unique (since A_0 arb., while rest of Fourier coeffs are uniquely determined).

This agrees with uniqueness thm, which guarantees that in case (ii) we've found all possible solutions

Example: BVP with mixed BCs

- Find $T(x,y)$ s.t. ① $\nabla^2 T = 0$ in $0<x<L$, $0<y<L$;
  ② $T(0,y)=0$, $T_x(L,y)=0$ for $0<y<L$;
  ③ $T(x,0)=0$, $T(x,L)=T^* \frac{x}{L}$ for $0<x<L$;

where $T^* \in \mathbb{R}^+$.

- $T = F(x)G(y)$ nontrivial $\Rightarrow -\frac{F''}{F} = \frac{G''}{G} = \omega^2$, $\omega \in \mathbb{R}^+$ wlog.

- ② & G nontrivial $\Rightarrow F(0)=0$, $F'(L)=0$

- Thus, $F = B\sin(\omega x)$ $(B\in\mathbb{R})$ & $B\neq 0, F'(L)=0 \Rightarrow \cos(\omega L)=0$
  $\Rightarrow \omega L = (n+\tfrac{1}{2})\pi$, $n\in\mathbb{N}$.

- Then $G'' = \omega^2 G$ & BC on $y=0$ $\Rightarrow G(y) \propto \sinh(\omega y)$

- Combo & superimpose $\Rightarrow T(x,y) = \sum_{n=1}^{\infty} B_n \sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right)\sinh\left((n+\tfrac{1}{2})\frac{\pi y}{L}\right)$
  $(B_n \in \mathbb{R})$

- BC on $y=L \Rightarrow T^*\frac{x}{L} = \underbrace{\sum_{n=0}^{\infty} B_n \sinh\left((n+\tfrac{1}{2})\pi\right)\sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right)}_{(+)}$ for $0<x<L$

- Assuming $\int\Sigma = \Sigma\int$,

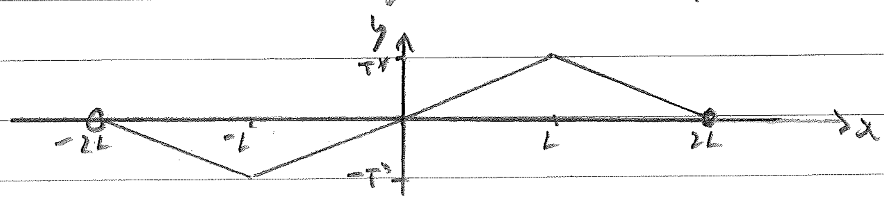
$$\int_0^L T^*\frac{x}{L}\sin\left((m+\tfrac{1}{2})\frac{\pi x}{L}\right)dx = \sum_{n=0}^{\infty} B_n\sinh\left((n+\tfrac{1}{2})\pi\right)\underbrace{\int_0^L \sin\left((m+\tfrac{1}{2})\frac{\pi x}{L}\right)\sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right)dx}_{\frac{L}{2}\delta_{mn}}$$

$$= B_m \sinh\left((m+\tfrac{1}{2})\pi\right)\frac{L}{2}$$

i.e. $B_n = \frac{2}{L\sinh\left((n+\frac{1}{2})\pi\right)}\int_0^L T^*\frac{x}{L}\sin\left((n+\tfrac{1}{2})\frac{\pi x}{L}\right)dx$ for $n\in\mathbb{N}$.

- Qn: To which fn does the generalized FS (+) converge for $x\in\mathbb{R}$?

Ans: Can use FCT to show it converges to the 4L-periodic function defined by



Wellposedness

- An IBVP or BVP is wellposed if $\exists!$ solution that depends continuously on the data in ICs and/or BCs.

Example : Wave Equation

- Suppose $y(x,t)$ s.t. ① $y_{tt} = y_{xx}$ for $-\infty < x < \infty$, $t > 0$,
  ② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

  where f and g are given.

- By D'Alembert's Formula $\exists!$ solution since ① - ② $\Rightarrow$

$$y(x,t) = \tfrac{1}{2}\left(f(x-t) + f(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s)\,ds.$$

- Now change data f, g to F, G and let Y be new solution:

$$Y(x,t) = \tfrac{1}{2}\left(F(x-t) + F(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} G(s)\,ds.$$

- Suppose $\exists\, \delta > 0$ s.t. $|f(x) - F(x)| < \delta$, $|g(x) - G(x)| < \delta$ $\forall x \in \mathbb{R}$. (†)

Then,
$$\begin{aligned} |y(x,t) - Y(x,t)| &= \left| \tfrac{1}{2}\left(f(x-t) - F(x-t)\right) + \tfrac{1}{2}\left(f(x+t) - F(x+t)\right) + \tfrac{1}{2}\int_{x-t}^{x+t} g(s) - G(s)\,ds \right| \\ &\leq \tfrac{1}{2}\left|f(x-t) - F(x-t)\right| + \tfrac{1}{2}\left|f(x+t) - F(x+t)\right| + \tfrac{1}{2}\int_{x-t}^{x+t} |g(s) - G(s)|\,ds \\ &\leq \tfrac{1}{2}\delta + \tfrac{1}{2}\delta + \tfrac{1}{2}\cdot 2t \cdot \delta \\ &= (1+t)\,\delta \qquad \text{for } -\infty < x < \infty,\ t \geq 0 \quad (\dagger\dagger) \end{aligned}$$

- Fix any $T > 0$ and any $\varepsilon > 0$.

  If we pick $\delta = \dfrac{\varepsilon}{1+T}$ in (†), then (††) implies

$$|y(x,t) - Y(x,t)| \leq \varepsilon \frac{1+t}{1+T} < \varepsilon \quad \text{for } -\infty < x < \infty,\ 0 < t < T$$

In this sense the solution depends continuously on the data and the IVP is well-posed.

Example: Try IVP for Laplace's equation!

- Suppose $y(x,t)$ s.t. ① $y_{xx} + y_{tt} = 0$ for $-\infty < x < \infty$, $t > 0$,
  ② $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

where f and g are given.

- Problem (I): $f_1 = 0$, $g_1 = 0 \Rightarrow y_1 = 0$ is a solution.

  Problem (II): $f_1 = 0$, $g_2 = \delta \cos\left(\frac{x}{\delta}\right) \Rightarrow y_2 = \delta^2 \cos\left(\frac{x}{\delta}\right)\sinh\left(\frac{t}{\delta}\right)$ is a solution for any $\delta > 0$.

- Observe that $|f_1(x) - f_2(x)| = 0$, $|g_1(x) - g_2(x)| \leq \delta$ $\forall x \in \mathbb{R}$.

  But $|y_1(0,t) - y_2(0,t)| = \delta^2 \sinh\left(\frac{t}{\delta}\right) \to \infty$ as $\delta \to 0+$ for any fixed $t > 0$, so cannot make $|y_1(0,t) - y_2(0,t)| < \varepsilon$ for all $0 < t < T$ by taking δ suitably small.

- IVP for Laplace's equation is not well-posed – called ill-posed!

Summary

1. Introduced theory of Fourier Series.
   - Periodic, even & odd functions & periodic extensions;
   - Euler's formulae for Fourier coeffs via orthogonality relations;
   - Statement of a powerful pointwise convergence thm;
   - Related rate of convergence to smoothness;
   - Discussed Gibb's phenomenon – try to avoid!

2. Heat equation
   - Derivation in 1D, 2D & 3D;
   - Simple solutions (fundamental & root cellar);

- units & nondimensionalization;
- Fourier's method for IBVPs;
- Generalized to inhomogeneous heat equation & BCs.
- Uniqueness.

3. Wave equation
- Derivation in 1D with gravity & air resistance;
- normal modes and natural frequencies;
- Fourier's method for IBVPs – plucked & flicked strings;
- Forced & damped wave equation with inhomogeneous BCs;
- normal modes for composite and weighted string;
- D'Alembert's solution and characteristic diagrams.
- Uniqueness.

4. Laplace's equation
- Derivation in 2D & 3D;
- Fourier's method for BVPs in (x, y) & (r, θ);
- Poisson's Integral Formula for Dirichlet problem on a disc;
- Uniqueness of Dirichlet problem;
- Nonexistence & nonuniqueness of Neumann problem

5. Well-posedness
- Introduced concepts developed later on in course.

- Problem sheet questions not too far from prelims questions, so should set you up well for exam.
- Should try at least 3-5 past papers, but maybe avoid the TT 2015 paper — it turned out much tougher than anticipated.
- Collect feedback.