M5 Fourier Series and PDEs

Course synopsis

Overview

While developing the theory of heat conduction in the early 19th century, Jean-Baptiste Joseph Fourier kick-started a mathematical revolution by claiming that "every" real-valued function defined on a finite interval could be expanded as an infinite series of elementary trigonometric functions — cosines and sines. The need for rigorous mathematical analysis to assess this astonishing claim led to a surprisingly large proportion of the material covered in prelims, part A and beyond (*e.g.* the definition of a function, the ε-δ definition of limit, the theory of convergence of sequences and series of functions, Lebesgue integration and Cantor's set theory). The implications of Fourier's claim for practical applications were no less powerful or far-ranging: the decomposition led to deep and fundamental insights into numerous physical phenomena (e.g. mass and heat transport, vibrations of elastic media, acoustics and quantum mechanics) and continue to be exploited today in numerous fields (e.g. signal processing, approximation theory and control theory).

In this course we introduce fundamental results for the pointwise convergence of Fourier's infinite trigonometric series — Fourier series. We then follow in Fourier's footsteps by using them to construct solutions to fundamental problems involving the heat equation, the wave equation and Laplace's equation — the three most ubiquitous partial differential equations in mathematics, science and engineering.

Reading list

[1] D. W. Jordan and P. Smith, *Mathematical Techniques* (Oxford University Press, 4th Edition, 2003)

[2] E. Kreyszig, *Advanced Engineering Mathematics* (Wiley, 10th Edition, 1999)

[3] G. F. Carrier and C. E. Pearson, *Partial Differential Equations — Theory and Technique* (Academic Press, 1988)

Synopsis (14 lectures)

Fourier series: Periodic, odd and even functions. Calculation of sine and cosine series. Simple applications concentrating on imparting familiarity with the calculation of Fourier coefficients and the use of Fourier series. The issue of convergence is discussed informally with examples. The link between convergence and smoothness is mentioned, together with its consequences for approximation purposes.

Partial differential equations: Introduction in descriptive mode on partial differential equations and how they arise. Derivation of (i) the wave equation of a string, (ii) the heat equation in one dimension (box argument only). Examples of solutions and their interpretation. D'Alembert's solution of the wave equation and applications. Characteristic diagrams (excluding reflection and transmission). Uniqueness of solutions of wave and heat equations.

PDEs with Boundary conditions. Solution by separation of variables. Use of Fourier series to solve the wave equation, Laplace's equation and the heat equation (all with two independent variables). Laplace's equation in Cartesian and in plane polar coordinates. Applications.

Authorship and acknowledgments

The author of these notes is **Jim Oliver**, taken largely from notes originally written Ruth Baker, Yves Capdeboscq, Alan Day and Janet Dyson, and typeset by **Benjamin Walker**.

Fourier Series & PDEs: Lectures 1-2

Motivation

Example: existence of a convergent Fourier series

- Recall
$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$
for $z \in \mathbb{C}$.

- If we let $z = e^{i\theta} = \cos\theta + i\sin\theta$, where $\theta \in \mathbb{R}$, then
$$\mathrm{Im}\,(e^z) = \mathrm{Im}\left(e^{\cos\theta} e^{i\sin\theta}\right) = e^{\cos\theta}\sin(\sin\theta)\,,$$
$$\mathrm{Im}\,(z^n) = \mathrm{Im}\left(e^{in\theta}\right) = \sin n\theta\,.$$

- Hence, $e^{\cos\theta}\sin(\sin\theta) = \underbrace{\sum_{n=1}^{\infty} \frac{\sin n\theta}{n!}}_{\text{Fourier (sine) series}}$ for $\theta \in \mathbb{R}$.

Example: heat conduction

- Suppose $T(x,t)$ is such that
  1. $T_t = T_{xx}$ for $0 < x < \pi\,,\ t > 0$,
  2. $T(0,t) = 0$, $T(\pi,t) = 0$ for $t > 0$,
  3. $T(x,0) = e^{\cos x}\sin(\sin x)$ for $0 < x < \pi$.

- Observe $T(x,t) = \sum_{n=1}^{N} b_N \sin(nx) e^{-n^2 t}$ satisfies (1) and (2) for all $b_1, b_2, \ldots, b_n \in \mathbb{R}$, $N \in \mathbb{N} \setminus \{0\}$.

- Question: how should we pick N and the constants b_n?

- Answer: $N = \infty$ and $b_N = \frac{1}{N!}$ to satisfy (3), i.e. a solution of the IBVP (1)–(3) is
$$T(x,t) = \sum_{n=1}^{\infty} \frac{1}{n!} \sin(nx) e^{-n^2 t}\,.$$

- But what about the other initial conditions?

Periodic, even and odd functions

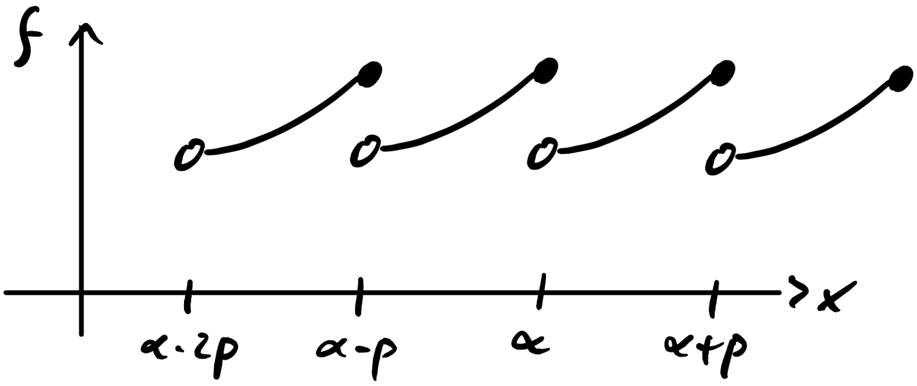
Definitions

- **Definition:** $f : \mathbb{R} \to \mathbb{R}$ is a periodic function if $\exists p > 0$ s.t. $f(x+p) = f(x) \forall x \in \mathbb{R}$. In this case p is a period for f and f is called **p–periodic**. A period is not unique, but if there exists a smallest such p it is called the prime period.

- Some examples:
  - $f =$ const. is p–periodic $\forall p > 0$, so has no prime period.

- $\sin x$ has prime period 2π.
- x and x^2 are not periodic.

- Note if f is periodic with prime period p then the graph of f repeats every p, e.g.



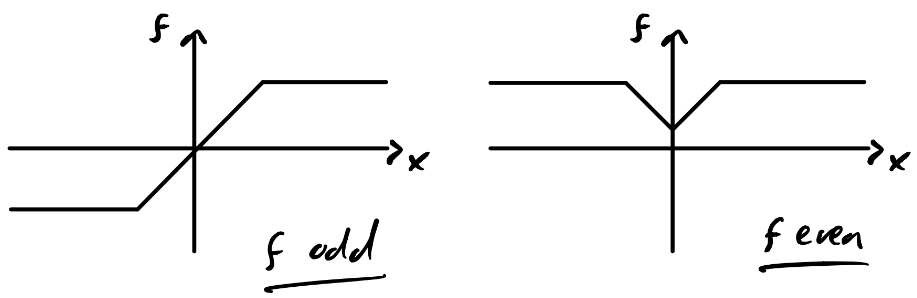
$f : (\alpha, \alpha + p] \to \mathbb{R}$ can be extended uniquely to be p–periodic.

- **Definition:** The periodic extension $F : \mathbb{R} \to \mathbb{R}$ of $f : (\alpha, \alpha+p] \to \mathbb{R}$ is defined by $F(x) = f(x - mp)$, where for each $x \in \mathbb{R}$, m is the unique integer such that $x - mp \in (\alpha, \alpha + p]$.

- f, g p–periodic implies:

1. f, g are np–periodic $\forall n \in \mathbb{N} \setminus \{0\}$,
2. $\alpha f + \beta g$ are p–periodic $\forall \alpha, \beta \in \mathbb{R}$,
3. fg is p–periodic,
4. $f(\lambda x)$ is p/λ–periodic $\forall \lambda > 0$,
5. $\int\limits_0^p f(x)\,\mathrm{d}x = \int\limits_\alpha^{\alpha+p} f(x)\,\mathrm{d}x \;\forall \alpha \in \mathbb{R}$.

- **Definition:** $f : \mathbb{R} \to \mathbb{R}$ is **odd** if $f(x) = -f(-x)\ \forall x \in \mathbb{R}$. Similarly, $f : \mathbb{R} \to \mathbb{R}$ is **even** if $f(x) = f(-x)\ \forall x \in \mathbb{R}$.
  - E.g. x^n is odd for n odd, and is even for n even (hence the naming convention).
  - Note symmetries of graphs of odd/even functions:



Properties of odd/even functions

If f, f_1 are odd and g, g_1 are even, then

1. $f(0) = 0$,
2. $\int\limits_{-\alpha}^{\alpha} f(x)\,\mathrm{d}x = 0 \;\forall \alpha \in \mathbb{R}$,

3. $\int\limits_{-\alpha}^{\alpha} g(x)\,\mathrm{d}x = 2\int\limits_{0}^{\alpha} g(x)\,\mathrm{d}x \;\forall x \in \mathbb{R}\,,$

4. fg odd, ff_1 even, and gg_1 even.

Fourier series for functions of period 2π

- Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π. We want an expansion for f of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) \qquad (\star)$$

- Q1: If $(\star)$ is true, can we find the constants a_n, b_n in terms of f?
- Q2: With these a_n and b_n, when is $(\star)$ true?

Question 1

- Suppose $(\star)$ is true and we can integrate it term by term, then

$$\int\limits_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 \int\limits_{-\pi}^{\pi} \mathrm{d}x + \sum_{n=1}^{\infty}\Bigg(a_n \underbrace{\int\limits_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x}_{0} + b_n \underbrace{\int\limits_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x}_{0}\Bigg)\,.$$

- Hence we have

$$a_0 = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x)\,\mathrm{d}x\,,$$

- I.e. $\frac{a_0}{2}$ is the mean of f over a period.
- **Lemma:** Let $m, n \in \mathbb{N}\setminus\{0\}$. Then we have the orthogonality relations:

$$\int\limits_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}$$

$$\int\limits_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x = 0$$

$$\int\limits_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn}\,,$$

  where δ_{mn} is Kronecker's delta, i.e.

$$\delta_{mn} = \begin{cases} 1 & \text{for } m = n\,, \\ 0 & \text{for } m \neq n\,. \end{cases}$$

  For proof see online lecture notes by Prof. Ruth Baker, in addition to the first problem sheet.

- Now, fix $m \in \mathbb{N}\setminus\{0\}$, multiply $(\star)$ by $\cos(mx)$ and assume that the integral of the infinite sum is the infinite sum of the integral. So

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x$$
$$+ \sum_{n=1}^{\infty}\left(a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x\right)$$
$$= \frac{1}{2}a_0 \cdot 0 + \sum_{n=1}^{\infty}(a_n \pi \delta_{mn} + b_n \cdot 0)$$
$$= \pi a_m \,.$$

- So

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x$$

  for $m \in \mathbb{N}\setminus\{0\}$.

- Similarly, we can fix $m \in \mathbb{N}\setminus\{0\}$, multiply ($\star$) by $\sin(mx)$ and assume that the integral of the infinite sum is the infinite sum of the integral to get

$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x$$

  for $m \in \mathbb{N}\setminus\{0\}$.

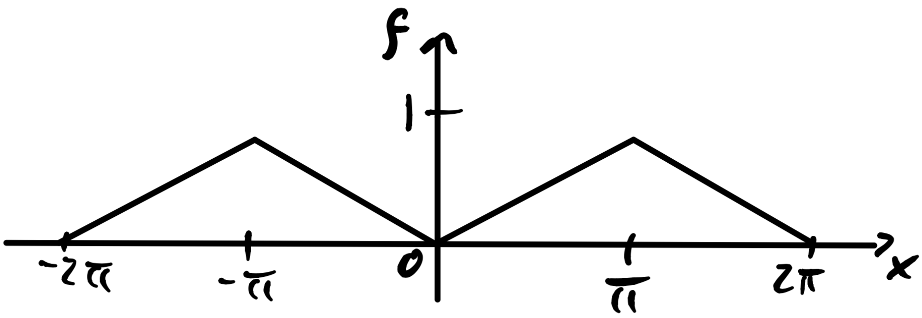
- **Definition:** Suppose f is such that the Fourier coefficients a_n and b_n as defined above exist for $n \in \mathbb{N}\setminus\{0\}$. Then we write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))\,,$$

  where $\sim$ means the RHS is the Fourier series for f, regardless of whether or not it converges to f.

- Note the factor of $\frac{1}{2}$ in the first term is for algebraic convenience.

Example 2.1. Find the Fourier series (FS) for the 2π-periodic function f defined by $f(x) = |x|$ for $-\pi < x \leq \pi$.



- $f(x)$ even, so $f(x)\cos(nx)$ is even and $f(x)\sin(nx)$ is odd. Thus

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,\mathrm{d}x\,, \quad b_n = 0\,.$$

- Calculate

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x\,\mathrm{d}x = \left[\frac{2}{\pi}\frac{x^2}{2}\right]_0^{\pi} = \pi\,.$$

- For $n > 0$ we use integration by parts:

$$(uv)' = u'v + uv' \implies [uv]_a^b = \int_a^b u'v + uv' \, \mathrm{d}x \, .$$

- Pick $u = x, v = \frac{1}{n}\sin(nx), a = 0, b = \pi$ to give

$$\left[\frac{x}{n}\sin(nx)\right]_0^\pi = \int_0^\pi 1 \cdot \frac{1}{n}\sin(nx) + x\cos(nx) \, \mathrm{d}x \, .$$

- So

$$\int_0^\pi x\cos(nx) \, \mathrm{d}x = -\int_0^\pi \frac{1}{n}\sin(nx) \, \mathrm{d}x = \left[\frac{\cos(nx)}{n^2}\right]_0^\pi = \frac{(-1)^n - 1}{n^2} \, ,$$

giving

$$a_n = -\frac{2}{\pi}\frac{[1-(-1)^n]}{n^2} = \begin{cases} 0 & \text{for } n = 2m, \ m \in \mathbb{N} \setminus \{0\} \, , \\ -\frac{4}{\pi(2m+1)^2} & \text{for } n = 2m+1, \ m \in \mathbb{N} \, . \end{cases}$$

- Hence,

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos((2m+1)x)}{(2m+1)^2} \, .$$

□

Remarks

1. Partial sums are defined by

$$S_N(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{N}\frac{\cos((2m+1)x)}{(2m+1)^2}$$

for $N \in \mathbb{N}$. Plots in the handout for Lecture 2 suggest that FS converges on $\mathbb{R}$, i.e.

$$\lim_{N\to\infty} S_N(x) = f(x) \quad \text{for } x \in \mathbb{R} \, . \qquad (\dagger)$$

2. If this is true, we can pick x to evaluate the sum of a series, e.g. $x = 0$ gives

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} \implies \sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8} \, .$$

Sine and cosine series

- Let f be 2π-periodic and such that the Fourier coefficients exist.
- If $f(x)$ is odd then

$$f(x)\cos(nx) \text{ is odd and } f(x)\sin(nx) \text{ is even}$$

$$\implies a_n = 0, \ b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin(nx) \, \mathrm{d}x$$

$$\implies f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx) \, , \text{ called a } \textbf{Fourier sine series.}$$

- Note that this is also true if f is odd only for $x \neq k\pi,\ k \in \mathbb{Z}$. Similarly, if $f(x)$ is even then

$$f(x) \sim \frac{a_0}{2} + \sim \sum_{n=1}^{\infty} a_n \cos(nx)\,, \text{ called a } \underline{\textbf{Fourier cosine series,}}$$

where

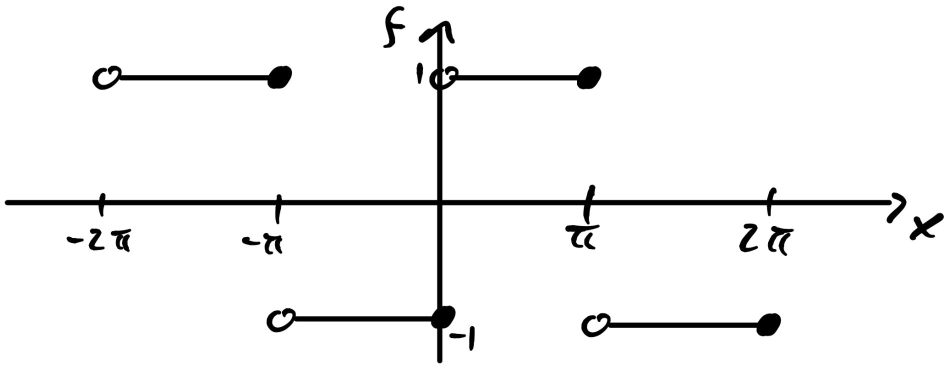
$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\, \mathrm{d}x\,.$$

Question 2

- When does the FS for f converge?

Example 2.2. Find the FS for the 2π-periodic function f defined by

$$f(x) = \begin{cases} 1 & \text{for } 0 < x \leq \pi\,, \\ -1 & \text{for } -\pi < x \leq 0\,. \end{cases}$$



- f is odd for $x \neq k\pi\ \ k \in \mathbb{Z}$, so

$$a_n = 0\,, \quad b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, \mathrm{d}x\,.$$

- $f(x) = 1$ for $0 < x < \pi$, so

$$b_n = \left[-\frac{2}{\pi} \frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2[1-(-1)^n]}{\pi n}$$

- Hence

$$f(x) \sim \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{2m+1}\,.$$

□

Remarks

1. Partial sums are defined by

$$S_N(x) = \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)x)}{2m+1} \qquad \text{for } N \in \mathbb{N}\,.$$

Plots in the handout for Lecture 2 suggest that

$$\lim_{N\to\infty} S_N(x) = \begin{cases} f(x) & \text{for } x \neq k\pi,\ k \in \mathbb{Z}\,, \\ 0 & \text{for } x = k\pi,\ k \in \mathbb{Z}\,. \end{cases} \qquad (\ddagger)$$

2. Note slower convergence than in example 2.1 and persistent overshoot near discontinuities of f - this is called Gibb's phenomenon (more to follow on this).

Convergence of Fourier series

Definition:

$$f(c_+) = \lim_{\substack{h\to 0\\ h>0}} f(c+h) \text{ if it exists (RH limit at c)}$$
$$f(c_-) = \lim_{\substack{h\to 0\\ h<0}} f(c+h) \text{ if it exists (LH limit at c)}$$

Remarks

1. f(x) need not be defined for $f(c_+)$ or $f(c_-)$ to exist.
2. Existence part is important, e.g. $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ implies that $f(0_\pm)$ do not exist.
3. $f(c_+) = f(c_-) = f(x) \iff f$ is continuous at c.
4. In example 2.2, f is continuous for $x \neq k\pi,\ k \in \mathbb{Z}$ with e.g. $f(0_+) = 1,\ f(0_-) = -1,\ f(\pi_+) = -1,\ f(\pi_-) = 1$.

Definition:

- f is **piecewise continuous** on $(a, b) \subseteq \mathbb{R}$ if there exists a finite number of points $x_1, \ldots, x_m$ with $a = x_1 < x_2 < \ldots < x_m = b$ such that
  - i) f is defined and continuous on (x_k, x_{k+1}) $\forall k = 1, \ldots, m-1$.
  - ii) $f(x_{k+})$ exists for $k = 1, \ldots, m-1$.
  - iii) $f(x_{k-})$ exists for $k = 2, \ldots, m$.
- Note that f need not be defined at its exceptional points $x_1, \ldots, x_m$!
- For example, the functions in examples 2.1 and 2.2 are piecewise continuous on any interval $(a, b) \subseteq \mathbb{R}$.

Theorem 2.1 (Fourier Convergence Theorem (FCT))**.** Let f be 2π-periodic, with f and f' piecewise continuous on $(-\pi, \pi)$. Then, the Fourier coefficients a_n and b_n exist, and

$$\frac{1}{2}\left(f(x_+) + f(x_-)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$

for $x \in \mathbb{R}$.

Note that FCT implies that (†) and (‡) are true.

Fourier Series & PDEs: Lectures 3-4

Remarks

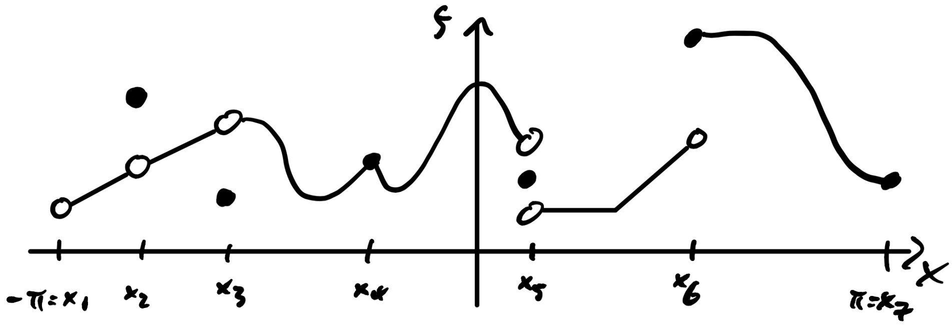
1. f, f' piecewise continuous (p.c.) on $(-\pi,\pi)$ $\implies$ $\exists x_1,\ldots,x_m \in \mathbb{R}$ with $-\pi = x_1 < x_2 < \ldots < x_m = \pi$ such that

   i) f and f' are continuous on (x_k, x_{k+1}) for $k = 1,\ldots,m-1$.

   ii) $f(x_{k+})$ and $f'(x_{k+})$ exist for $k = 1,\ldots,m-1$.

   iii) $f(x_{k-})$ and $f'(x_{k-})$ exist for $k = 2,\ldots,m$.

   Thus, in any period f, f' are continuous except possibly at a finite number of points. At each such point f' need not be defined, and one or both of f and f' may have a jump discontinuity. E.g.



   E.g.

$$f(x) = \begin{cases} x^{1/2} & \text{for } 0 < x \le \pi\,, \\ 0 & \text{for } -\pi < x \le 0 \end{cases} \implies f'(x) = \begin{cases} \frac{1}{2}x^{-1/2} & \text{for } 0 < x < \pi\,, \\ 0 & \text{for } -\pi < x < 0\,, \\ \text{undefined} & \text{for } x = 0, \pi\,. \end{cases}$$

   Thus f p.c. on $(-\pi,\pi)$, but f' is not.

2. Proof not examinable, but one method is as follows:
   Firstly, show that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) - \frac{1}{2}\left(f(x_+) + f(x_-)\right)$$

$$= \int_0^{\pi} F(x,t) \sin\left[\left(N + \frac{1}{2}\right)t\right] \mathrm{d}t\,, \quad (2.1)$$

   where

$$F(x,t) = \frac{1}{\pi}\left(\frac{f(x+t) - f(x_+)}{t} + \frac{f(x-t) - f(x_-)}{t}\right)\left(\frac{t}{2\sin(t/2)}\right)\,.$$

   Secondly, show $F(x,t)$ is a p.c. function of t on $(0,\pi)$, so that the Riemann-Lebesgue Lemma (Analysis III) implies

$$\int_0^{\pi} F(x,t) \sin\left[\left(N + \frac{1}{2}\right)t\right] \mathrm{d}t \to 0 \qquad \text{as } N \to \infty\,.$$

3. f continuous at x $\implies$ $\frac{1}{2}\left(f(x_+) + f(x_-)\right) = f(x)$.

4. If f defined only on e.g. $(-\pi,\pi]$, FCT holds for its 2π-periodic extension.

5. Can integrate termwise under weaker conditions, e.g. if f is only 2π-periodic and p.c. on $(-\pi, \pi)$, then FCT implies

$$\int_0^x f(x)\,\mathrm{d}x = \frac{1}{2}a_0 x + \sum_{n=1}^{\infty}\left(a_n \int_0^x \cos(nx)\,\mathrm{d}x + b_n \int_0^x \sin(nx)\,\mathrm{d}x\right)$$

for $x \in \mathbb{R}$. Note that LHS is 2π-periodic iff $a_0 = 0$.

6. But we need stronger conditions to differentiate termwise, e.g. if f is 2π-periodic and continuous on $\mathbb{R}$ with f' and f'' p.c. on $(-\pi, \pi)$, then FTC implies

$$\frac{1}{2}\left(f'(x_+) + f'(x_-)\right) = \sum_{n=1}^{\infty}\left(a_n \frac{\mathrm{d}}{\mathrm{d}x}(\cos(nx)) + b_n \frac{\mathrm{d}}{\mathrm{d}x}(\sin(nx))\right)$$

for $x \in \mathbb{R}$.

Rate of convergence

- The smoother f, i.e. the more continuous derivatives it has, the faster the convergence of the FS for f.
- If the first jump discontinuity is in the p^{th} derivative of f, with the convension that $p = 0$ if there is a jump discontinuity in f, then typically the non-zero a_n and b_n decay like $1/n^{p+1}$ as $n \to \infty$. For example, $p = 1$ in example 2.1, while $p = 0$ in example 2.2.
- This is an extremely useful result in practice (e.g. how many terms to keep for an accurate approximation) and for checking calculations. For example, for approximately 1% accuracy we need 100 terms for $p = 0$, and only 10 terms for $p = 1$.

Gibb's phenomenon

- This is the persistent overshoot in example 2.2 near a jump discontinuity. It happens whenever a jump discontinuity exists.
- As the number of terms in the partial sum tends to ∞, the width of the overshoot region tends to 0 (by FCT), whilst the height of the overshoot region approaches $\gamma|f(x_+) - f(x_-)|$, where in the case of example 2.2 we have

$$\gamma = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\sin x}{x}\,\mathrm{d}x \approx 1.18 \approx 9\%\,.$$

- This is awful for approximation purposes!

Functions of any period

- Suppose now that $f(x)$ is a periodic function of period $2L > 0$.
- Make the transformation $x = \frac{LX}{\pi}$, $f(x) = g(X)$, then for $X \in \mathbb{R}$

$$g(X + 2\pi) = f\left(\frac{L}{\pi}(X + 2\pi)\right) = f\left(\frac{LX}{\pi} + 2L\right) = f\left(\frac{LX}{\pi}\right) = f(X)\,.$$

- Thus, g is 2π-periodic and we can use transformation to derive theory for f from that for g above.
- Here we summarise the key results.

Fourier Series

$$g(X) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right)$$
$$\implies f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Fourier coefficients

$$a_n := \frac{1}{\pi} \int_{-\pi}^{\pi} g(X) \cos(nX)\, \mathrm{d}X = \frac{1}{\pi} \int_{-L}^{L} g\left(\frac{\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \frac{\pi}{L}\, \mathrm{d}x$$
$$= \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x\,.$$

Similarly,

$$b_n := \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x\,.$$

Important remark

These formulae may be derived directly from the FS for f by assuming that $\int \sum = \sum \int$, and using the orthogonality relations

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = L\delta_{mn}\,,$$
$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = 0\,,$$
$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = L\delta_{mn}\,,$$

where $n, m \in \mathbb{N} \setminus \{0\}$.

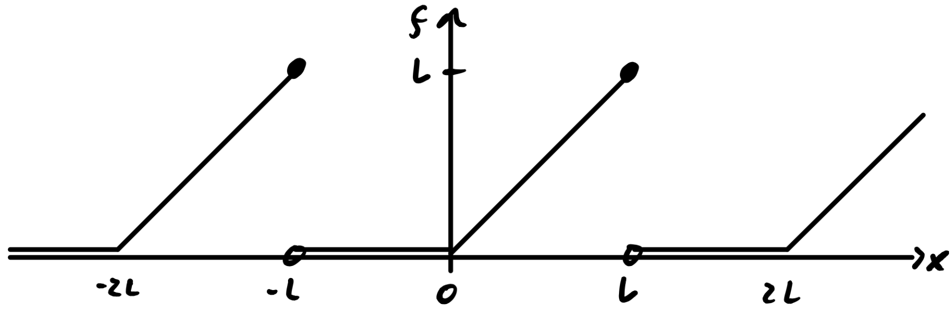
Theorem 2.2 (Fourier Convergence Theorem (FCT))**.** Let f be $2L$-periodic with f and f' p.c. on $(-L, L)$. Then a_n and b_n exist, and

$$\frac{1}{2}\left(f(x_+) + f(x_-)\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

for $x \in \mathbb{R}$.

Example 2.3. Find the FS of the $2L$-periodic function f defined by

$$f(x) = \begin{cases} x & \text{for } 0 < x \leq L\,, \\ 0 & \text{for } -L < x \leq 0\,. \end{cases}$$



- We have

$$a_n = \frac{1}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x\,, \quad b_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\,.$$

- We find $a_0 = \frac{1}{L}\frac{L^2}{2} = \frac{L}{2}$, but for $n > 0$ it is a bit quicker to evaluate

$$\begin{aligned}
a_n + ib_n &= \frac{1}{L}\int_0^L \underbrace{x}_{u}\ \underbrace{\exp\left(\frac{in\pi x}{L}\right)}_{v'}\mathrm{d}x \\
&= \left[\frac{1}{L}\underbrace{x}_{u}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\right]_0^L - \frac{1}{L}\int_0^L \underbrace{1}_{u'}\ \underbrace{\frac{L}{in\pi}\exp\left(\frac{in\pi x}{L}\right)}_{v}\mathrm{d}x \\
&= \frac{L}{in\pi}\exp(in\pi) - \left[\frac{1}{L}\left(\frac{L}{in\pi}\right)^2\exp\left(\frac{in\pi x}{L}\right)\right]_0^L \\
&= \frac{iL(-1)^{n+1}}{n\pi} + \frac{L}{n^2\pi^2}\left((-1)^n - 2\right)\,.
\end{aligned}$$

- Thus

$$f(x) \sim \frac{L}{4} + \sum_{n=1}^{\infty}\left(-\frac{2L}{(2n-1)^2\pi^2}\cos\left(\frac{(2n-1)\pi x}{L}\right) + \frac{L(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right)$$

- FCT implies that the FS converges to $f(x)$ for $x \neq (2k+1)L,\ k\in\mathbb{Z}$, and to $\frac{1}{2}\left(f(L_+) + f(L_-)\right) = \frac{1}{2}(0+L) = \frac{L}{2}$ otherwise.

- For example

$$\begin{aligned}
x = 0 \implies 0 = f(0) = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} &\implies \sum_{k=0}^{\infty}\frac{1}{(2k+1)^2} = \frac{\pi^2}{8}\,, \\
x = L \implies \frac{L}{2} = \frac{L}{4} - \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{-1}{(2n-1)^2} &\implies \text{the same sum!}
\end{aligned}$$

Cosine and sine series

- Suppose now $f : [0, L] \to \mathbb{R}$ is given. Periodic extension of period $2L$ is not unique, but there are two especially useful ones for PDE applications.

- **Definition:** The even/odd $2L$-periodic extensions, f_e and f_o respectively, of $f : [0, L] \to \mathbb{R}$ are defined by

$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L\,, \\ f(-x) & \text{for } -L < x < 0\,, \end{cases} \qquad \text{with } f_e(x+2L) = f_e(x) \text{ for } x\in\mathbb{R}$$

and

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L\,, \\ -f(-x) & \text{for } -L < x < 0\,, \end{cases} \qquad \text{with } f_o(x+2L) = f_o(x) \text{ for } x\in\mathbb{R}$$

– Note that $f_o(x)$ is odd for $x \neq kL,\ k \in \mathbb{Z}$, and odd (on $\mathbb{R}$) iff $f(0) = f(L) = 0$.

- <u>**Definition:**</u> The Fourier cosine and sine series for $f : [0, L] \to \mathbb{R}$ are the Fourier series for f_e and f_o respectively, i.e.

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \text{ where } a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x\,,$$

$$f_o(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right), \text{ where } a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x\,,$$

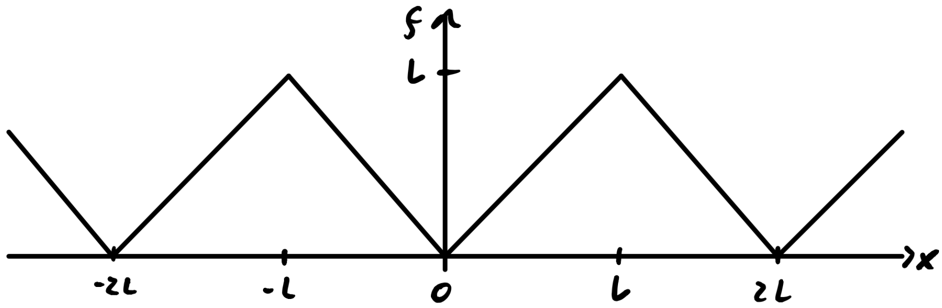
  – Note that if f is continuous on $[0, L]$ and f' p.c. on $(0, L)$, then FCT gives

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f_e(x) \text{ for } x \in \mathbb{R}\,,$$

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} f_o(x) & \text{for } x \neq kL,\ k \in \mathbb{Z}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.4. Find the cosine and sine series of $f : [0, L] \to \mathbb{R}$ defined by $f(x) = x$ for $0 \leq x \leq L$.

$$f_e(x) = \begin{cases} x & \text{for } 0 \leq x \leq L\,, \\ -x & \text{for } -L < x < 0\,, \end{cases} \text{, i.e. } f_e(x) = |x| \text{ for } -L < x \leq L\,.$$



- We have

$$a_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x\,,$$

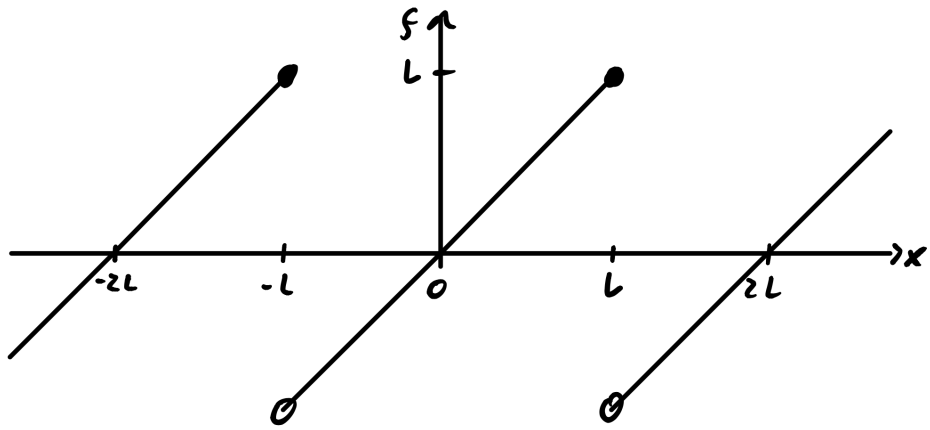
  so that

$$f_e(x) \sim \underbrace{\frac{L}{2} - \sum_{n=0}^{\infty} \frac{4L}{(2n+1)^2\pi^2} \cos\left(\frac{(2n+1)\pi x}{L}\right)}_{\text{Cosine series}} = f_e(x)$$

  by the FCT.

- Similarly,

$$f_o(x) = \begin{cases} x & \text{for } 0 \leq x \leq L\,, \\ -(-x) & \text{for } -L < x < 0\,, \end{cases} \text{, i.e. } f_o(x) = x \text{ for } -L < x \leq L\,.$$



- We have

$$b_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x\,,$$

so that

$$f_o(x) \sim \underbrace{\sum_{n=0}^{\infty} \frac{2L(-1)^{n+1}}{n\pi} \sin\left(\frac{n\pi x}{L}\right)}_{\text{Sine series}} = \begin{cases} f_o(x) & \text{for } x \neq kL,\ kL \in \mathbb{Z}\,, \\ 0 & \text{otherwise.} \end{cases} \,.$$

□

Remarks

1. $f_e + f_o = 2f_{Ex.\ 2.3} \implies FS(f_e) + FS(f_o) = FS(2f_{Ex.\ 2.3}$.
2. Rates of convergence? $p = 1$ for f_e, and $p = 0$ for f_o, as predicted.
3. Question: Which truncated series gives the best approximation to f on $[0, L]$?
   Answer: Cosine series since

   i) it converges everywhere on $[0, L]$;

   ii) it converges more rapidly;

   iii) it does not exhibit Gibb's phenomena.

Fourier Series & PDEs: Lectures 5-6

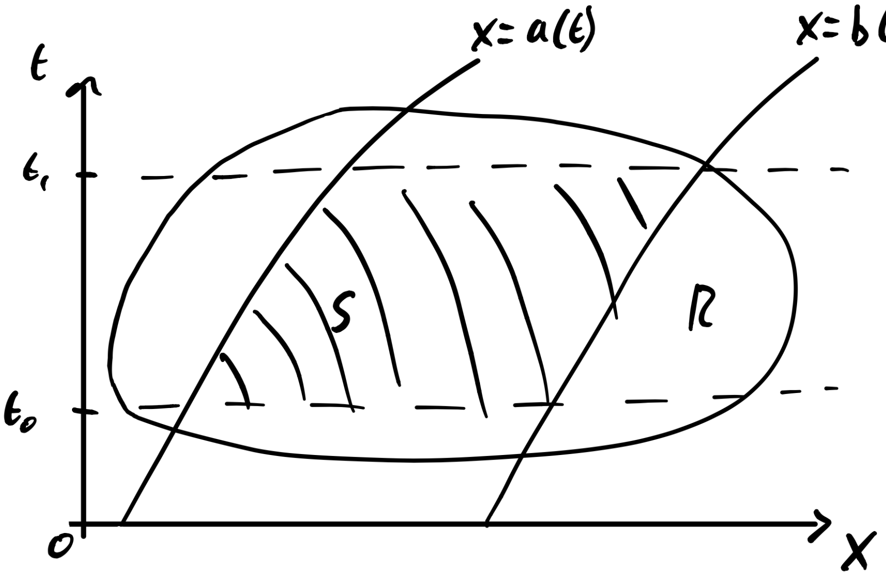
The PDEs we shall study

PDE	Name	Unknown	Parameters
$T_t = \kappa T_{xx}$	Heat equation	$T(x,t)$	$\kappa > 0$
$y_{tt} = c^2 y_{xx}$	Wave equation	$y(x,t)$	$c > 0$
$T_{xx} + T_{yy} = 0$	Laplace's equation	$T(x,y)$	None

- We shall derive them using physical principles and develop methods to solve several physically important problems formed by imposing appropriate BCs and/or ICs - different for each of them!

Some preliminaries

- **Leibniz's Integral Rule (LIR)**



If F, F_t are continuous on $R \supseteq S$ and $a, \dot{a}, b, \dot{b}$ are continuous for $t \in [t_0, t_1]$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{a(t)}^{b(t)} F(x,t)\,\mathrm{d}t = \int_{a(t)}^{b(t)} F_t(x,t)\,\mathrm{d}x + F(b(t),t)\dot{b}(t) - F(a(t),t)\dot{a}(t)\,.$$

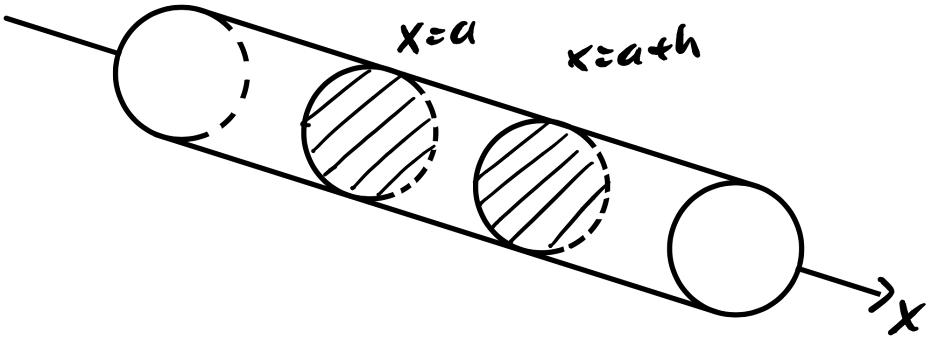
Note: a, b constant $\implies \frac{\mathrm{d}}{\mathrm{d}t} \int_a^b F(x,t)\,\mathrm{d}x = \int_a^b F_t(x,t)\,\mathrm{d}x$.

- **Lemma:** $f(x)$ continuous $\implies \frac{1}{h} \int_a^{a+h} f(x)\,\mathrm{d}x \to f(a)$ as $h \to 0$.

The heat equation

Derivation in 1D

- Consider a straight rigid isotropic conducting rod (e.g. metal) with insulated lateral surfaces lying along the x-axis.



- We'll need the following quantities:

Symbol	Quantity	SI units
x	Axial distance	m
t	Time	s
$T(x,t)$	Temperature	K
$q(x,t)$	Heat flux in +ve x-direction	$\mathrm{J\,m^{-2}\,s^{-1}}$ (1J=1N m)
A	Cross-sectional area	$\mathrm{m^2}$
ρ	Rod density	$\mathrm{kg\,m^{-3}}$
c	Rod specific heat	$\mathrm{J\,kg^{-1}\,K^{-1}}$
k	Rod thermal conductivity	$\mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}}$
κ	Rod thermal diffusivity	$\mathrm{m^2\,s^{-1}}$

- Conservation of energy in fixed section $a \le x \le a+h$:

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\left(A\int_a^{a+h}\rho C T\,\mathrm{d}x\right)}_{(1)} = \underbrace{Aq(a,t)}_{(2)} - \underbrace{Aq(a+h,t)}_{(3)}\ .$$

  (1) is the time rate of change of internal energy in $a \le x \le a+h$.

  (2) is the rate at which heat enters through $x = a$.

  (3) is the rate at which heat leaves through $x = a+h$.

- Note this is also true for $h < 0$ with appropriate reinterpretation.
- Assuming T_t is continuous, LIR with $a, a+h$ constant gives

$$\frac{\rho c}{h}\int_a^{a+h} T_t\,\mathrm{d}x + \frac{q(a+h,t)-q(a,t)}{h} = 0\,.$$

- Assuming q_x is continuous and taking the limit as $h \to 0$, the above lemma gives

$$\rho c T_t + q_x = 0\,. \qquad (\dagger)$$

Fourier's Law

- This is the constitutive law

$$q = -kT_x\,. \qquad (\ddagger)$$

- Models flow of heat from high to low temperatures.
- ($\dagger$) and ($\ddagger$) $\implies$ $\rho c T_t - (kT_x)_x = 0$, or

$$\boxed{T_t = \kappa T_{xx}}$$

  where $\kappa = \frac{k}{\rho c}$.

- Note we assumed T_t and $q_x = -kT_{xx}$ to be continuous.

Units and nondimensionalisation

- **Notation**

  $[p]$ is the dimension of p in fundamental dimensions (M, L, T, Θ etc) or e.g. SI units (kg, m, s, K).

- Both sides of an equation modelling a physical process must have same dimensions, e.g. $[(1)] = [(2)] = [(3)] = \mathrm{J\,s^{-1}}$.

- Exploit to check solutions are dimensionally correct and to determine dimensions of parameters, e.g.

$$[k] = \frac{[q]}{[T_x]} = \frac{\mathrm{J\,m^{-2}\,s^{-1}}}{\mathrm{K\,m^{-1}}} = \mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}}\,, \quad [\kappa] = \frac{[T_t]}{[T_{xx}]} = \frac{[x^2]}{[t]} = \mathrm{m^2\,s^{-1}}\,.$$

- **Nondimensionalisation**

  Method of scaling variables with typical values to derive dimensionless equations. These usually contain dimensionless parameters that characterise the relative importance of the physical mechanisms in the model.

Example (IBVP). • Suppose $T(x,t)$ such that

(1) $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;

(2) $T(0,t) = T_0$, $T(L,t) = T_1$ for $t > 0$;

(3) $T(x,0) = T_2 \frac{x}{L}\left(1 - \frac{x}{L}\right)$ for $0 < x < L$.

- Five dimensional parameters: κ, L, T_0, T_1, T_2.

- Nondimensionalise by scaling $x = L\hat{x}$, $t = L^2\hat{t}/\kappa$, and $T(x,t) = T_2\hat{T}(\hat{x},\hat{t})$, where L^2/κ is the timescale for diffusive transport of heat.

- Chain rule $\implies$

$$\frac{\partial T}{\partial t} = T_2 \frac{\partial \hat{T}}{\partial \hat{t}} \frac{\mathrm{d}\hat{t}}{\mathrm{d}t} = \frac{\kappa T_2}{L^2} \frac{\partial \hat{T}}{\partial \hat{t}}\,,$$
$$\frac{\partial T}{\partial x} = T_2 \frac{\partial \hat{T}}{\partial \hat{x}} \frac{\mathrm{d}\hat{x}}{\mathrm{d}x} = \frac{T_2}{L} \frac{\partial \hat{T}}{\partial \hat{x}} \quad \text{etc.}$$

- Hence (1)-(3) $\implies$ dimensionless problem for $\hat{T}(\hat{x},\hat{t})$ given by

  (1′) $\hat{T}_{\hat{t}} = \hat{T}_{\hat{x}\hat{x}}$ for $0 < \hat{x} < 1$, $\hat{t} > 0$;

  (2′) $\hat{T}(0,\hat{t}) = \alpha_0$, $\hat{T}(1,\hat{t}) = \alpha_2$ for $\hat{t} > 0$;

  (3′) $\hat{T}(\hat{x},0) = \hat{x}(1-\hat{x})$ for $0 < \hat{x} < 1$.

- Two dimensionless parameters $\alpha_0 = \frac{T_0}{T_2}$, $\alpha_1 = \frac{T_1}{T_2}$.

- If $\hat{T} = \hat{T}(\hat{x},\hat{t};\alpha_0,\alpha_1)$ is a solution of (1′)-(3′), then a solution of (1)-(3) is given by

$$\frac{T}{T_2} = \hat{T}\left(\frac{x}{L}, \frac{\kappa t}{L^2}; \frac{T_0}{T_2}, \frac{T_1}{T_2}\right)\,,$$

  i.e. T/T_2 must be a function of x/L and $\kappa t/L^2$!

Heat conduction in a finite rod

- Consider IBVP for $T(x,t)$ given by

  (1) $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;

  (2) $T(0,t) = 0$, $T(L,t) = 0$ for $t > 0$;

  (3) $T(x,0) = f(x)$ for $0 < x < L$.

  where the initial temperature profile $f(x)$ is given.

- Solve using **Fourier's method**:

  (I) Use method of separation of variables to find the countably infinite set of nontrivial separable solutions satisfying the PDE (1) and BCs (2), each containing an arbitrary constant.

  (II) Use the principle of superposition - that the sum of any number of solutions of a linear problem is also a solution (assuming convergence) - to form the general series solution that is the infinite sum of the separable solutions of the PDE and BCs.

  (III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the IC (3).

Remarks

1. (1) and (2) are linear since, if T_1 and T_2 satisfy them, then so does $\alpha T_1 + \beta T_2$ $\forall \alpha, \beta \in \mathbb{R}$.
2. To verify resulting series is actually a solution of the PDE, need it to converge sufficiently rapidly that T_t and T_{xx} can be computed by termwise differentiation - we largely gloss over such issues.

Step I

- $T = F(x)G(t) \underset{(1)}{\implies} FG' = \kappa F''G \implies \frac{F''}{F} = \frac{G'}{\kappa G}$, $(FG \neq 0)$.
- LHS independent of t and RHS independent of $x \implies$ LHS = RHS independent of x and t. Thus LHS = RHS = $-\lambda$, say, $\lambda \in \mathbb{R}$.
- Hence

$$\boxed{-F''(x) = \lambda F(x) \text{ for } 0 < x < L} \qquad (\dagger)$$

- (2) $\implies F(0)G(t) = 0$ and $F(L)G(t) = 0$ for $t > 0$.
  T nontrivial $\implies G$ nontrivial $\implies$

$$\boxed{F(0) = 0,\ F(L) = 0} \qquad (\ddagger)$$

- Now need to find all $\lambda \in \mathbb{R}$ such that ODE BVP ($\dagger$)-($\ddagger$) for $F(x)$ has a nontrivial solution. Consider cases

  (i) $\lambda = -\omega^2$, $(\omega > 0$ wlog$)$

  $(\dagger) \implies F'' - \omega^2 F = 0 \implies F = A\cosh\omega x + B\sinh\omega x$, $(A, B \in \mathbb{R})$.
  $(\ddagger) \implies A = 0, B\sinh\omega L = 0 \implies F = 0$.

  (ii) $\lambda = 0$

  $(\dagger) \implies F'' = 0 \implies F = A + Bx$, $(A, B \in \mathbb{R})$.
  $(\ddagger) \implies A = 0, BL = 0 \implies F = 0$.

  (iii) $\lambda = \omega^2$, $(\omega > 0$ wlog$)$

  $(\dagger) \implies F'' + \omega^2 F = 0 \implies F = A\cos\omega x + B\sin\omega x$, $(A, B \in \mathbb{R})$.
  $(\ddagger) \implies A = 0, B\sin\omega L = 0$. But $B \neq 0$ for F nontrivial, so $\sin\omega L = 0$, so $\omega L = n\pi$, $n \in \mathbb{N} \setminus \{0\}$.

- For $\lambda = \omega^2 = \left(\frac{n\pi}{L}\right)^2$, $F = B\sin\left(\frac{n\pi x}{L}\right)$ and $F \propto \exp\left(-\kappa\left(\frac{n\pi}{L}\right)^2 t\right)$.
- Hence, nontrivial separable solutions given by

$$T_n(x,t) = b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

where n is a positive integer and b_n a constant.

Step II

- Since (1)-(2) are linear, formally the principle of superposition implies that the general series solution is given by

$$T(x,t) = \sum_{n=1}^{\infty} T_n(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right).$$

Step III

- IC (3) can only be satisfied if

$$f(x) = T(x,0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L\,.$$

- The theory of FS $\implies$ the Fourier coefficients are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \text{ for } n \in \mathbb{N}\setminus\{0\}\,, \qquad (\ddagger\ddagger)$$

which determine the b_n and hence a solution.

Remarks

(1) f, f' piecewise continuous on $(0, L)$ $\implies$ sine series converges to $\frac{1}{2}(f(x_+) + f(x_-))$ for $x \in (0, L)$ and to 0 for $x = 0, L$, so can deal with jump discontinuities in ICs.

(2) In questions often asked to derive ($\ddagger\ddagger$) via orthogonality relations rather than quoting it. The relevant ones here are

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x = \frac{L}{2}\delta_{mn}\,,$$

where $m, n \in \mathbb{N}\setminus\{0\}$. Assuming $\int\sum = \sum\int$, then gives, for $n \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned}
\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x &= \frac{2}{L}\int_0^L \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \sum_{m=1}^{\infty} b_m \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \sum_{m=1}^{\infty} b_m \delta_{mn} \\
&= b_m\,.
\end{aligned}$$

Example 3.1.

$$f(x) = \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{2\pi x}{L}\right).$$

Then $b_1 = 1$, $b_2 = \frac{1}{2}$, $b_n = 0$ otherwise.

Example 3.2.

$$f(x) = \begin{cases} T^\star & \text{for } L_1 < x < L_2\,, \\ 0 & \text{otherwise.} \end{cases}$$

- Then

$$b_n = \frac{2}{L}\int_{L_1}^{L_2} T^\star \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2T^\star}{n\pi}\left(\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right)$$

- We've found a solution (assuming sufficiently rapid convergence), but is it the only solution?

Fourier Series & PDEs: Lectures 7-8

Uniqueness

Theorem 3.1. The IBVP has only one solution.

Proof: Suppose $T, \tilde{T}$ are solutions and let $W = T - \tilde{T}$. By linearity, (1)-(3) $\implies$

(1′) $W_t = T_t - \tilde{T}_t = \kappa T_{xx} - \kappa\tilde{T}_{xx} = \kappa(T - \tilde{T})_{xx} = \kappa W_{xx}$ for $0 < x < L$, $t > 0$;

(2′) $W = T - \tilde{T} = 0$ at $x = 0, L$ for $t > 0$;

(3′) $W(x, 0) = T(x, 0) - \tilde{T}(x, 0) = f(x) - f(x) = 0$ for $0 < x < L$.

Strategy: deduce that $W(x, t) \equiv 0$.

Trick: analyse $I(t) := \frac{1}{2}\int_0^L W(x,t)^2 \, dx$.

- Evidently $I(t) \geq 0$ for $t \geq 0$ and $I(0) = 0$ by (3′).
- But

$$\begin{aligned}
\frac{dI}{dt} &= \int_0^L W W_t \, dx && \text{(by LIR)}\\
&= \int_0^L W \kappa W_{xx} \, dx && \text{(by (1′))}\\
&= [\kappa W W_x]_0^L - \kappa \int_0^L W_x W_x \, dx && \text{(by IBP)}\\
&= -\kappa \int_0^L W_x^2 \, dx && \text{(by (2′))}\\
&\leq 0\,,
\end{aligned}$$

- So $I(t)$ cannot increase!
- Hence, $0 \leq I(t) \leq I(0) = 0$, giving $I(t) = 0$ for $t \geq 0$, so that $W = 0$ and $T = \tilde{T}$ for $0 \leq x \leq L$, $t \geq 0$ (assuming continuity of W there). □
- Note that this method of proof works for any linear BCs for which $[WW_x]_0^L \leq 0$, e.g. the radiative BCs $W_x(0, t) = -\alpha W(0, t)$, $W_x(L, t) = \alpha W(L, t)$ for $t > 0$, where α is a positive parameter.

Non-zero steady state

Example 3.3. Solve the IBVP

(1) $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;

(2) $T(0, t) = T_0$, $T(L, t) = T$ for $t > 0$;

(3) $T(x, 0) = 0$ for $0 < x < L$,

where T_0, T_1 are prescribed constants.

- We cannot use separation of variables straight away because the BCs are not homogeneous (unless $T_0 = T_1 = 0$).

- Conjecture that $T(x,t) \to S(x)$ as $t \to \infty$, where $S(x)$ is the steady-state solution of (1)-(2), so that

$$0 = \kappa S_{xx} \text{ for } 0 < x < L, \text{ with } S(0) = T_0,\ S(L) = T)1\,.$$

- Thus, $S(x) = T_0\left(1 - \frac{x}{L}\right) + T_1\left(\frac{x}{L}\right)$, a linear temperature profile.
- Now let $T(x,t) = S(x) + U(x,t)$, then (1)-(3) $\implies U(x,t)$ satisfies the IBVP

  (1′) $(S+U)_t = \kappa(S+U)_{xx} \implies U_t = \kappa U_{xx}$ for $0 < x < L$, $t > 0$;

  (2′) $S(0) + U(0,t) = T_0$, $S(L) + U(L,t) = T_1 \implies U(0,t) = 0$, $U(L,t) = 0$ for $t > 0$;

  (3′) $S(x) + U(x,0) = 0 \implies U(x,0) = -S(x)$ for $0 < x < L$.

- We solved this problem last lecture using Fourier's method:

$$U(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

  where

$$b_n = -\frac{2}{L}\int_0^L S(x)\sin\left(\frac{n\pi x}{L}\right) = -\frac{2}{n\pi}(T_0 - (-1)^n T_1)\,.$$

- Note that T_0, T_1 in BCs (2) end up in IC (3′) - sometimes called "method of shifting the data".

Other BCs

Example 3.4. Solve the IBVP

(1) $T_t = \kappa T_{xx}$ for $0 < x < L$, $t > 0$;

(2) $T_x(0,t) = 0$, $T_x(L,t) = 0$ for $t > 0$;

(3) $T(x,0) = f(x)$ for $0 < x < L$,

- Note both ends thermally insulated since $q = -kT_x = 0$ at $x = 0, L$.
- Apply Fourier's method on problem sheet 4 to give

$$T(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

  where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Remarks

1. The constant separable solution $T = \frac{a_0}{2}$ of (1)-(2) comes from the case in which the separation constant is zero.
2. As $t \to \infty$, $T(x,t) \to \frac{a_0}{2} = \frac{1}{L}\int_0^L f(x)\,\mathrm{d}x$, i.e. mean of the initial temperature.
3. Uniqueness by a similar argument to before.

Inhomogeneous PDE and BCs

Example 3.5. Solve the IBVP

(1) $\rho C T_t = k T_{xx} + Q(x,t)$ for $0 < x < L$, $t > 0$;

(2) $T_x(0,t) = \phi(t)$, $T_x(L,t) = \psi(t)$ for $t > 0$;

(3) $T(x,0) = f(x)$ for $0 < x < L$,

where $Q(x,t), \phi(t), \psi(t)$ and $f(x)$ are given.

- Note Q is volumetric heat source (e.g. due to radiation of chemical reaction) and heat flux in positive x-direction $q = -kT_x$.
- Now both PDE and BCs are inhomogeneous!
- Deal first with BCs by shifting the data.
- Find $S(x,t)$ such that $S_x(0,t) = \phi(t), S_x(L,t) = \psi(t)$ for $t > 0$, e.g.

$$S(x,t) = -\phi(t)\frac{(x-L)^2}{2L} + \psi(t)\frac{x^2}{2L}\,.$$

- Let $T(x,t) = S(x,t) + U(x,t)$, then (1)-(3) $\implies$ $U(x,t)$ satisfies the IBVP

  (1′) $\rho C U_t = k U_{xx} + \tilde{Q}(x,t)$ for $0 < x < L$, $t > 0$;

  (2′) $U_x(0,t) = 0$, $U_x(L,t) = 0$ for $t > 0$;

  (3′) $U(x,0) = \tilde{f}(x)$ for $0 < x < L$,

  where

$$\left.\begin{aligned} \tilde{Q}(x,t) &= Q(x,t) + kS_{xx} - \rho c S_t \\ \tilde{f}(x) &= f(x) - S(x,0) \end{aligned}\right\} \text{Known in terms of } Q, \phi, \psi \text{ and } f.$$

- If $\tilde{Q} = 0$, then can solve (1′)-(3′) via Fourier's method as in example 3.4.
- This suggests we seek a solution of the form

$$U(x,t) = \frac{U_0(t)}{2} + \sum_{n=1}^{\infty} U_n(t)\cos\left(\frac{n\pi x}{L}\right), \tag{†}$$

  where the functions $U_0(t), U_1(t), \ldots$ are TBD.

- Since (†) is a Fourier cosine series, its Fourier coefficients are given by

$$U_n(t) = \frac{2}{L}\int_0^L U(x,t)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \text{ for } n \in \mathbb{N}\,.$$

- We can then use (1′)-(3′) to derive ODEs for the U_n as follows.
- By Leibniz's integral rule

$$\begin{aligned} \rho c\frac{\mathrm{d}U_n}{\mathrm{d}t} &= \frac{2}{L}\int_0^L \rho c U_t \cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ &= \frac{2}{L}\int_0^L (kU_{xx} + \tilde{Q})\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x && \text{(by (1′))} \\ &= \frac{2k}{L}\int_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x + \tilde{Q}_n(t)\,, \end{aligned}$$

where

$$\tilde{Q}_n(t) = \frac{2}{L} \int\limits_0^L \tilde{Q}(x,t) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

are the know coefficients of the Fourier cosine series for $\tilde{Q}$.

- How do we deal with the U_{xx} integral? IBP twice via

$$(uv' - u'v)' = uv'' - u''v \implies [uv' - u'v]_a^b = \int\limits_a^b uv'' - u''v \, \mathrm{d}x \, .$$

- Let $u = U$, $v = \cos\left(\frac{n\pi x}{L}\right)$, $a = 0$, $b = L$, then

$$\underbrace{\left[U\left(-\frac{n\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) - U_x \cos\left(\frac{n\pi x}{L}\right)\right]_0^L}_{=0 \text{ by } (2')} = \int\limits_0^L U\left(-\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right)\right) - U_{xx}\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

$$\implies \frac{2}{L}\int\limits_0^L U_{xx}\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}\frac{2}{L}\int\limits_0^L U\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x = -\frac{n^2\pi^2}{L^2}U_n \, .$$

- Hence $\rho c \frac{\mathrm{d}U_n}{\mathrm{d}t} + \frac{kn^2\pi^2}{L^2}U_n = \tilde{Q}(t)$ for $t > 0$.

- IC? $(3') \implies U_n(0) = \frac{2}{L}\int\limits_0^L \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$.

Remarks

(1) Reduced problem to a countably infinite set of ODEs - recover solution of example 3.4 when $Q = 0, \phi = 0, \psi = 0$.

(2) Can solve explicitly for the U_n using an integrating factor.

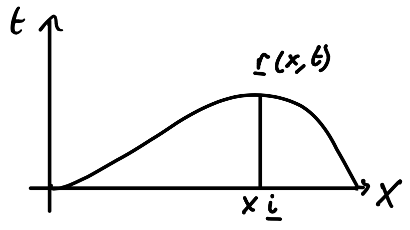
(3) Uniqueness proof the same as for example 3.4.

The wave equation

Derivation in 1D

- Consider the small transverse vibrations of a homogeneous extensible elastic string stretched initially along the x-axis at time $t = 0$.

- A point at $x\mathbf{i}$ at time $t = 0$ is displaced to $\mathbf{r}(x,t) = x\mathbf{i} + y(x,t)\mathbf{j}$ at time $t > 0$, where the transverse displacement $y(x,t)$ is TBD.

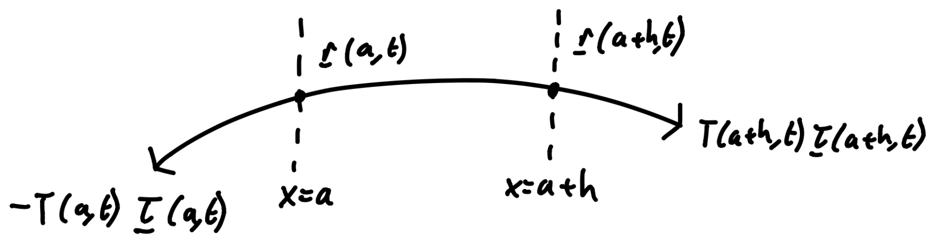


- Consider piece of string in a fixed region $a \le x \le a + h$.

- Linear momentum is $\int_a^{a+h} \rho \mathbf{r}_t \, \mathrm{d}x$, where ρ is constant line density of the string ($[\rho] = \mathrm{kg\,m^{-1}}$).

- Assuming no resistance to bending (cf a ruler), the string to the right of $\mathbf{r}(x,t)$ exerts at this point a force $T(x,t)\tau(x,t)$ on the string to the left, where $T(x,t)$ is the tension ($[T] = \mathrm{N} = \mathrm{kg\,m\,s^{-2}}$) and $\tau = \mathbf{r}_x / |\mathbf{r}_x|$ is the unit tangent vector in the +ve x-direction.

- Assuming tension so large that gravity and air resistance are negligible, the forces on the string in $a \le x \le a+h$ are



- NII says $\frac{\mathrm{d}}{\mathrm{d}t}$ (linear momentum) = net force, so

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{a+h} \rho \mathbf{r}_t \, \mathrm{d}x\right) = T(a+h,t)\tau(a+h,t) - T(a,t)\tau(a,t)\,.$$

- Assuming $\mathbf{r}_{tt}$ is continuous, LIR then gives

$$\frac{1}{h}\int_a^{a+h} \rho \mathbf{r}_{tt} \, \mathrm{d}x = \frac{T(a+h,t)\tau(a+h,t) - T(a,t)\tau(a,t)}{g}\,.$$

- Assuming $(T\tau)_x$ is continuous, let $h \to 0$ (from above and below)

$$\implies \rho \mathbf{r}_{tt} = \frac{\partial}{\partial x}(T\tau)$$
$$\implies \rho y_{tt}\mathbf{j} = \frac{\partial}{\partial x}\left(\frac{T\mathbf{i} + Ty_x\mathbf{j}}{(1+y_x^2)^{\frac{1}{2}}}\right)\,.$$

- Now small displacement $\implies$ small slope $\implies$ $|y_x| \ll 1$

$$\implies \left(1 + y_x^2\right)^{\frac{1}{2}} = 1 + \frac{1}{2}(y_x)^2 + \cdots$$

$\implies$ to a first approximation (i.e. neglecting quadratic and higher order terms)

$$\rho y_{tt}\mathbf{j} = T_x\mathbf{i} + (Ty_x)_x\mathbf{j}\,.$$

- x-direction $\implies$ $T_x = 0$ $\implies$ $T = T(t)$, i.e. tension is spatially uniform, but could vary with t, e.g. tuning a guitar string. We shall assume T =constant, which is the case in many practical applications.

- y-direction $\implies$ $\rho y_{tt} = (Ty_x)_x = Ty_{xx}$

- We have derived the **wave equation**

$$\boxed{y_{tt} = c^2 y_{xx}}$$

where $c = \sqrt{\frac{T}{\rho}}$ is the **wave speed** (for reasons that will become apparent).

Units and nondimensionalisation

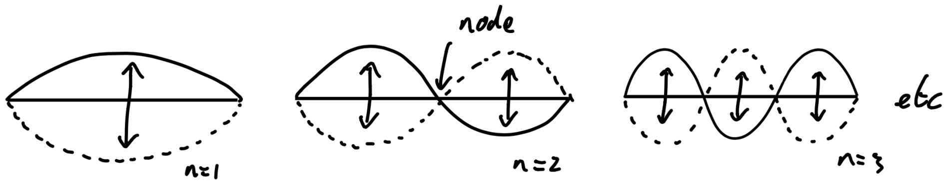
- $\left[c^2\right] = \frac{[y_{tt}]}{[y_{xx}]} = \frac{\mathrm{m\,s^{-2}}}{\mathrm{m\,m^{-2}}} = \mathrm{m^2\,s^{-2}} \implies [c] = \mathrm{m\,s^{-1}}$.
- Check: $\left[c^2\right] = \frac{[T]}{[\rho]} = \frac{\mathrm{N}}{\mathrm{kg\,m^{-1}}} = \frac{\mathrm{kg\,m\,s^{-2}}}{\mathrm{kg\,m^{-1}}} = \mathrm{m^2\,s^{-2}}$ ✓.
- On what timescale does a displacement travel a distance L? Scale $x = L\hat{x}, t = t_0\hat{t}, y = H\hat{y}$

$$\implies \frac{H}{t_0^2}\hat{y}_{\hat{t}\hat{t}} = \frac{Hc^2}{L^2}\hat{y}_{\hat{x}\hat{x}} \implies \hat{y}_{\hat{t}\hat{t}} = \hat{y}_{\hat{x}\hat{x}}$$

provided $t_0 = \frac{L}{c}$.

Normal modes of vibration for a finite string

- Suppose string stretched between $x = 0$ and $x = L$ and the ends held fixed.
- Slinky experiment suggests there exists discrete modes of vibration:



- To analyse mathematically, we seek separable solutions to
  (1) $y_{tt} = c^2 y_{xx}$ for $0 < x < L$, $t \in \mathbb{R}$;
  (2) $y(0,t) = 0$, $y(L,t) = 0$, $t \in \mathbb{R}$.
- $y = F(x)G(t)$ in (1) $\implies FG'' = c^2F''G \implies \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2G(t)}$, assuming $FG \neq 0$.
- LHS independent of t and RHS independent of $x \implies$ LHS = RHS independent of x and t. Thus LHS = RHS = $-\lambda \in \mathbb{R}$, say.
- Hence $-F'' = \lambda F$ for $0 < x < L$ (I)
- (2) and G nontrivial $\implies F(0) = 0$, $F(L) = 0$ (II)
- $\lambda \leq 0 \implies$ (I)-(II) have only the trivial solution $F = 0$.
- Let $\lambda = \omega^2$, with $\omega > 0$ wlog.
- (I) $\implies F = A\cos\omega x + B\sin\omega x$, $(A, B \in \mathbb{R})$.
- (II) $\implies A = 0$, $B\sin\omega L = 0$.
- F nontrivial $\implies B \neq 0 \implies \sin\omega L = 0 \implies \omega L = n\pi$, $n \in \mathbb{N}\setminus\{0\}$.
- $\omega = \frac{n\pi}{L} \implies F(x) = B\sin\left(\frac{n\pi x}{L}\right)$, $G(t) = C\cos\left(\frac{n\pi ct}{L}\right) + D\sin\left(\frac{n\pi ct}{L}\right)$.
- Combo $\implies$ normal modes (nontrivial separable solutions of (1)-(2)) are

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n\cos\left(\frac{n\pi ct}{L}\right) + b_n\sin\left(\frac{n\pi ct}{L}\right)\right)$$

or

$$y_n(x,t) = c_n\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi c}{L}(t + \epsilon_n)\right),$$

where $a_n, b_n \in \mathbb{R}$, $c_n, \epsilon_n \in \mathbb{R}$ for each $n \in \mathbb{N}\setminus\{0\}$.

Remarks

(1) y_n periodic in t with prime period $p = \frac{2\pi}{n\pi c/L} = \frac{2L}{nc}$ and frequency (or pitch) $\frac{1}{p} = \frac{nc}{2L}$.

(2) y_1 is the fundamental mode; $\frac{c}{2L}$ the fundamental frequency; all other modes have a frequency that is an integer multiple of $\frac{c}{2L}$.

(3) Consistent with slinky experiment.

(4) Normal modes are an example of a standing wave since $y = f(x) \times$ oscillatory function.

[Next time: use Fourier's method to solve IBVP obtained by imposing two ICs.]

Fourier Series & PDEs: Lectures 9-10

IBVP for a finite string

- Find $y(x,t)$ such that
  (1) $y_{tt} = c^2 y_{xx}$ for $0 < x < L$, $t > 0$;
  (2) $y(0,t) = 0$, $y(L,t) = 0$ for $t > 0$;
  (3) $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $0 < x < L$.
- Use Fourier's method [f, g are initial transverse displacement and velocity]

Step I: Find all nontrivial separable solutions of (1)-(2)

- Last time we found that these (normal modes) are

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right),$$

where $a_n, b_n \in \mathbb{R}$ and $n \in \mathbb{N} \setminus \{0\}$.

Step II: Formally apply the principle of superposition

- (1)-(2) are linear, so superimpose the normal modes (assuming convergence) to obtain the general series solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t).$$

Step III: Use theory of FS to satisfy the ICs

- (3) can only be satisfied if

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L,$$
$$g(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right) \text{ for } 0 < x < L.$$

- Assuming $\int \sum = \sum \int$, we deduce that

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x}_{=\frac{L}{2}\delta_{nm}} = \frac{L}{2} a_m$$

$$\implies a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \text{ and similarly}$$
$$b_n \frac{n\pi c}{L} = \frac{2}{L} g(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

Example 4.1.

$$f(x) = A \sin\left(\frac{\pi x}{L}\right) + B \sin\left(\frac{2\pi x}{L}\right) \implies a_1 = A,\ a_2 = B, \text{ and the rest zero.}$$

Example 4.2.

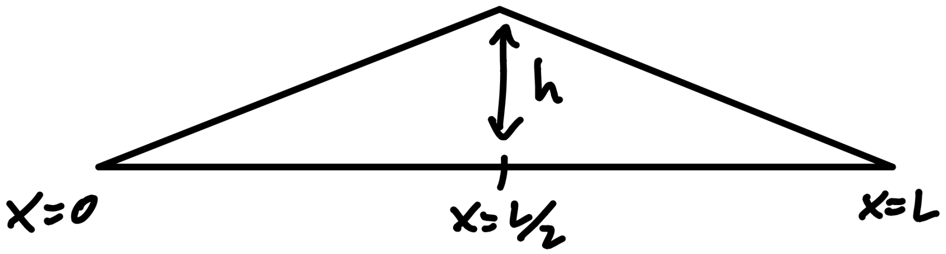
$$f(x)=0,\ g(x)=V_0\sin^3\left(\frac{\pi x}{L}\right) \implies a_n=0\forall n\,.$$

Trick: $\sin^3\left(\frac{\pi x}{L}\right)=\frac{3}{4}\sin\left(\frac{\pi x}{L}\right)-\frac{1}{4}\sin\left(\frac{3\pi x}{L}\right)$

$$\implies \frac{\pi c}{L}=\frac{3V_0}{4},\ b_2=0,\ \frac{3\pi c}{L}b_3=-\frac{V_0}{4} \text{ and rest zero.}$$

Example 4.3. (Guitar string)

$$f(x)=\begin{cases} 2hx/L & \text{for } 0\le x\le L/2\,,\\ 2h(L-x)/L & \text{for } L/2\le x\le L\,,\end{cases} \qquad g(x)=0\,.$$



$$a_n=\frac{2}{L}\int_0^{L/2}\frac{2hx}{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x+\frac{2}{L}\int_{L/2}^{L}\frac{2h(L-x)}{L}\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x=\frac{8h}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$b_n=0\,.$$

Since

$$\sin\left(\frac{n\pi}{2}\right)=\begin{cases} 0 & \text{for } n=2m, m\in\mathbb{N}\setminus\{0\}\,,\\ (-1)^m & \text{for } n=2m+1, m\in\mathbb{N}\end{cases}\,,$$

we find

$$y(x,t)=\frac{8h}{\pi^2}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^2}\sin\left(\frac{(2m+1)\pi x}{L}\right)\cos\left(\frac{(2m+1)\pi ct}{L}\right).$$

Example 4.4. (Piano string)

$$f(x)=0,\quad g(x)=\begin{cases} v & \text{for } L_1\le L\le L_2\,,\\ 0 & \text{otherwise.}\end{cases}$$

$$a_n=0 \text{ and } b_n=\frac{L}{n\pi c}\frac{2}{L}\int_{L_1}^{L_2} v\sin\left(\frac{n\pi x}{L}\right)$$

$$\implies y(x,t)=\frac{2vL}{c\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[\cos\left(\frac{n\pi L_2}{L}\right)-\cos\left(\frac{n\pi L_1}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$$

Energy and uniqueness

- Consider IBVP (1)-(3).
- KE of string is $\int_0^L \frac{1}{2}\rho\,|\mathbf{r}_t|^2\,\mathrm{d}x=\int_0^L\frac{1}{2}\rho y_t^2\,\mathrm{d}x$.
- Elastic PE of string is product of tension and extension, i.e.

$$T\left(\int_0^L |\mathbf{r}_x|\,\mathrm{d}x-L\right)=T\int_0^L (1+y_x^2)^{\frac{1}{2}}-1\,\mathrm{d}x$$

- But $|y_x| \ll 1$, so $\left(1+y_x^2\right)^{\frac{1}{2}} - 1 = \frac{1}{2}y_x^2 + \cdots$, so to a first approximation (neglecting cubic and higher order terms), the elastic PE is $\int_0^L \frac{1}{2} T y_x^2 \, \mathrm{d}x$.
- Hence the energy of a string is

$$E(t) = \int_0^L \frac{1}{2}\rho y_t^2 + \frac{1}{2} T y_x^2 \, \mathrm{d}x \, .$$

Lemma: : If y satisfies (1)-(2), then $E(t)$ is constant for $t > 0$. Proof:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \int_0^L \rho y_t y_{tt} + T y_x y_{xt} \, \mathrm{d}x && \text{(by LIR)} \\
&= \int_0^L T y_t y_{xx} + T y_x y_{xt} \, \mathrm{d}x && \text{(by (1))} \\
&= \int_0^L (T y_t y_x)_x \, \mathrm{d}x \\
&= [T y_t y_x]_{x=0}^{x=L} \\
&= 0 \, ,
\end{aligned}$$

since (2) $\implies y_t(0,t) = y_t(L,t) = 0$ for $t > 0$.

Remarks

(1) Lemma and (3) $\implies E(t) = E(0) = \int_0^L \frac{1}{2}\rho (g(x))^2 + \frac{1}{2} T (f'(x))^2 \, \mathrm{d}x$.

(2) Lemma $\implies$ Energy in nth normal mode is given by

$$E_n(t) = E_n(0) = \int_0^L \frac{1}{2}\rho \left(y_{nt}(x,0)\right)^2 + \frac{1}{2} T \left(y_{nx}(x,)\right)^2 \, \mathrm{d}x = \frac{n^2\pi^2 T}{4L}(a_n^2 + b_n^2) \, .$$

(3) Can then use Parseval's Identity for g and f' to show that $E(0) = \sum_{n=1}^{\infty} E_n(0)$, i.e. total energy is sum of energy in each normal mode (which are constant throughout motion and set by ICs by remark (2)).

Theorem 4.1 (Uniqueness)**.** The IBVP has at most one solution.

Proof: Let $W(x,t) = y - \tilde{y}$, where $y, \tilde{y}$ are two solutions. Then, by linearity,

(1′) $W_{tt} = c^2 W_{xx}$ for $0 < x < L$, $t > 0$;

(2′) $W(0,t) = 0$, $W(L,t) = 0$ for $t > 0$;

(3′) $W(x,0) = 0$, $W_t(x,0) = 0$ for $0 < x < L$.

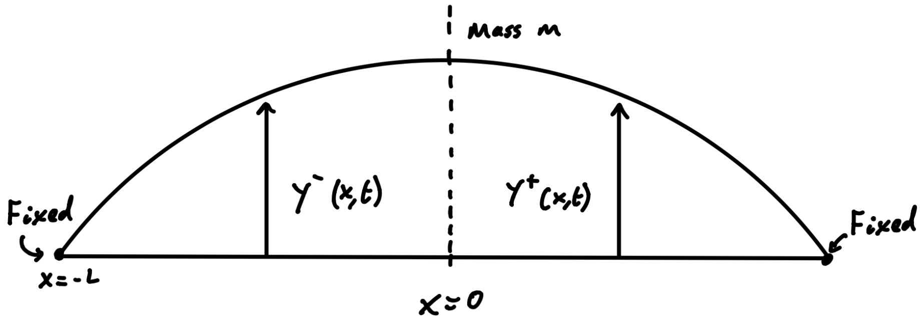
The above lemma applied to W gives

$$\int_0^L \frac{\rho}{2}(W_t)^2 + \frac{T}{2}(W_x)^2 \, \mathrm{d}x = E(t) \underset{(1')\text{-}(2')}{=} E(0) \underset{(3')}{=} 0 \text{ for } t \geq 0$$

$\implies W_t = W_x = 0$ for $0 < x < L, \; t > 0$ (assuming W_t, W_x continuous there)
$\implies W =$ constant for $0 < x < L, \; t > 0$
$\implies W = 0$ for $0 \leq x \leq L, \; t \geq 0$ by (2′) or (3′), (assuming W continuous for $0 \leq x \leq L, t \geq 0$)

Normal modes for a weighted string

- Setup:



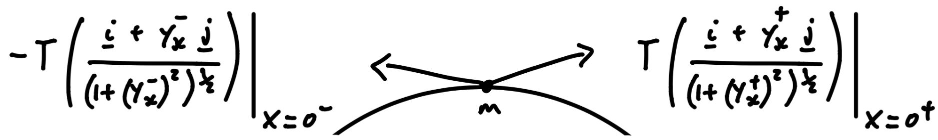
- What are the normal modes?
- PDEs:

$$(1^-)\ y^-_{tt} = c^2 y^-_{xx} \text{ for } -L < x < 0$$
$$(1^+)\ y^+_{tt} = c^2 y^+_{xx} \text{ for } 0 < x < L$$

- BCs:

$$(2^-)\ y^-(-L,t) = 0$$
$$(2^+)\ y^+(L,t) = 0$$
$$(3)\ y^-(0,t) = y^+(0,t) = Y(t), \text{ say.}$$

- $Y(t)$ TBD so need a second BC at $x = 0$ via NII for the mass.
- Forces on mass (neglecting gravity and air resistance):



- Small transverse displacement $\implies |y^{\pm}_x| \ll 1 \implies (1+(y^{\pm}_x)^2)^{\frac{1}{2}} = 1+$h.o.t, so to a first approximation mass remains on y-axis (because x-force components balance), while in y-direction

$$(4)\ m\ddot{Y} = T\left(y^+_x|_{x=0^+} - y^-_x|_{x=0^-}\right)$$

- Separate variables: $y^{\pm} = F_{\pm}(x)G(t)$

$$(1^{\pm}) \implies \frac{F''_{\pm}(x)}{F_{\pm}(x)} = \frac{G''(t)}{c^2 G(t)} = -\lambda \in \mathbb{R}, \text{ say, assuming } F_{\pm}G \neq 0$$
$$(2^{\pm})\ G \neq 0 \implies F_-(-L) = 0,\ F_+(L) = 0 \qquad (a^{\pm})$$
$$(3)\ G \neq 0 \implies F_-(0) = F_+(0) \qquad (b)$$
$$(4)\ G \neq 0 \implies mF_{\pm}(0)G''(t) = T(F'_+(0_+) - F_-(0_-))G(t)$$
$$\underset{c^2=\frac{T}{\rho}}{\implies} -\frac{\lambda m}{\rho}F_{\pm}(0) = F'_+(0_+) - F'_-(0_-) \qquad (c)$$

- Can show $\lambda \leq 0 \implies F_{\pm} = 0$. Let $\lambda = \omega^2$, $\omega > 0$ wlog.

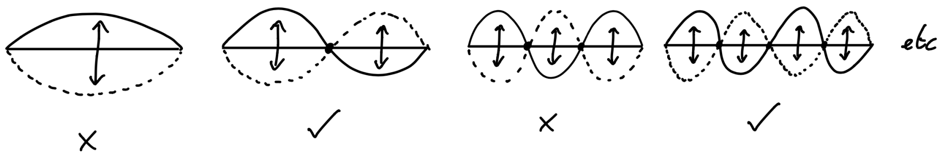
- Then

$$F_-'' + \omega^2 F_- = 0 \text{ for } -L < x < 0\,,$$
$$F_+'' + \omega^2 F_+ = 0 \text{ for } 0 < x < L\,.$$

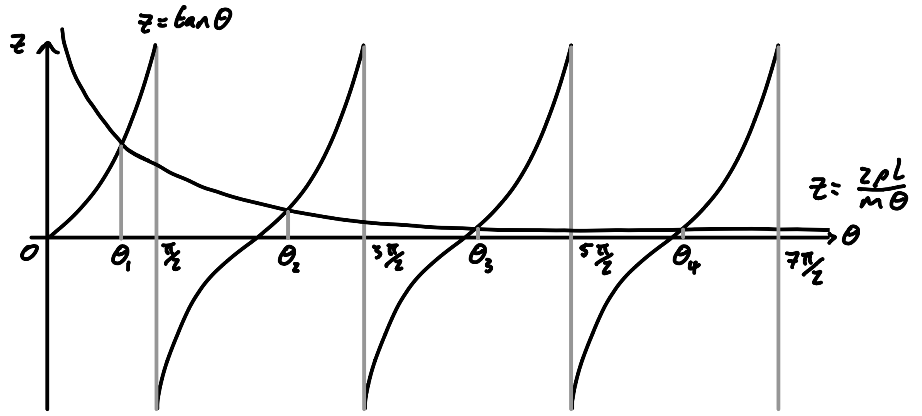
$$(a^\pm) \implies F_- = A\sin\omega(L+x),\ F_+B\sin\omega(L-x) \quad (A,B\in\mathbb{R}) \tag{4.1}$$

$$(b)-(c) \implies \underbrace{\begin{bmatrix} \sin\omega L & -\sin\omega L \\ \cos\omega L - \frac{m\omega}{\rho}\sin\omega L & \cos\omega L \end{bmatrix}}_{M} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{†}$$

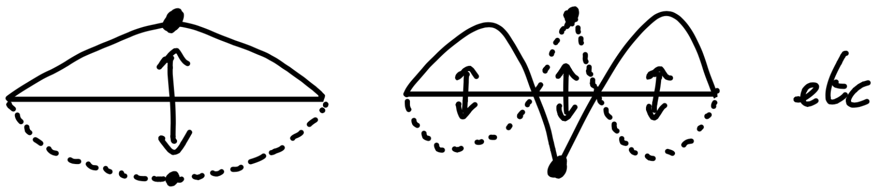
- $\begin{bmatrix} A \\ B \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \det M = 0 \implies \sin\omega L\left(2\cos\omega L - \frac{m\omega}{\rho}\sin\omega L\right) = 0.$
- Hence, either
  - (i) $\sin\omega L = 0 \implies \omega = \frac{n\pi}{L},\ n\in\mathbb{N}\setminus\{0\}$
  - (ii) $\cot\omega L = \frac{m\omega}{2\rho} \implies \cot\theta = \frac{m\theta}{2\rho L}$, where $\theta = \omega L$.
  - (iii) In each case, $G'' + \omega^2c^2G = 0 \implies G(t) = C\cos(\omega ct+\epsilon)$, $(c\epsilon\in\mathbb{R})$, where wlog $C = 1$.
  - (iv) In case (i), (†) $\implies A = -B \implies \begin{cases} y_- = A\sin\omega(L+x)\cos(\omega ct+\epsilon) \\ y_+ = -A\sin\omega(L-x)\cos(\omega ct+\epsilon) \end{cases}$
- This means that the normal modes are the same as for a string of length $2L$ with a node at $x = 0$, i.e. mass stationary.



- In case (ii) there are infinitely many roots $\theta_1 < \theta_2 < \cdots$
- To see, plot LHS and RHS of e.g. $\tan\theta = \frac{2\rho L}{m\theta}$



- Hence, infinitely many normal modes $\omega_n = \frac{\theta_n}{L}$, $n\in\mathbb{N}\setminus\{0\}$.
- Now (†) $\implies A = B \implies \begin{cases} y_- = A\sin\omega(L+x)\cos(\omega ct+\epsilon) \\ y_+ = A\sin\omega(L-x)\cos(\omega ct+\epsilon) \end{cases}$
- This means that the normal modes are symmetric about $x = 0$:



- Try with slinky!

General solution to the wave equation

- Remarkable fact: can write down all solutions of $y_{tt} = c^2 y_{xx}$!
- Let $y(x,t) = Y(\xi, \eta)$, $\xi = x - ct$, $\eta = x + ct$ (as in Introductory Calculus).

$$
\begin{aligned}
\implies y_x &= Y_\xi \xi_x + Y_\eta \eta_x = Y_\xi + Y_\eta \\
y_{xx} &= (Y_\xi + Y_\eta)_\xi \xi_x + (Y_\xi + Y_\eta)_\eta \eta_x = Y_{\xi\xi} + 2Y_{\xi\eta} + Y_{\eta\eta} \\
y_t &= Y_\xi \xi_t + Y_\eta \eta_t = -cY_\xi + cY_\eta \\
y_{tt} &= (-cY_\xi + cY_\eta)_\xi \xi_t + (-cY_\xi + cY_\eta)_\eta \eta_t = c^2(Y_{\xi\xi} - 2Y_{\xi\eta} + Y_{\eta\eta})
\end{aligned}
$$

where we assumed $Y_{\xi\eta} = Y_{\eta\xi}$.

- Hence, $y_{tt} - c^2 y_{xx} = -4c^2 Y_{\xi\eta}$
- Wave equation, $c > 0$ gives

$$
\begin{aligned}
&\implies Y_{\xi\eta} = 0 \\
&\implies Y_\xi = F'(\xi)\,, \text{ say} \\
&\implies (Y - F(\xi))_\xi = 0 \\
&\implies Y - F(\xi) = G(\eta)\,, \text{ say} \\
&\implies y = F(x - ct) + G(x + ct)
\end{aligned}
$$

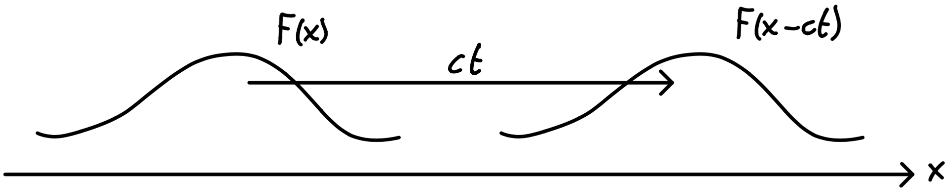
where F, G are arbitrary twice-continuously-differentiable functions.

Remarks

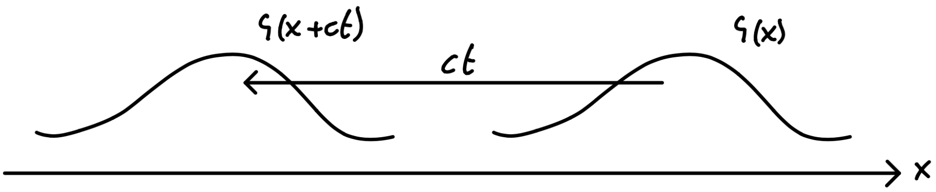
(1) c.f. # of arbitrary constants in general solution to a second order ODE.

(2) Easy to verify this is a solution (see supplementary notes): we've shown that all solutions must be of this form.

(3) $F(x - ct)$ is a travelling wave of constant shape moving to the right with speed c:



$G(x + ct)$ is a travelling wave of constant shape moving to the left with speed c:



Waves on an infinite string: D'Alembert's formula

- Consider the IVP

  (1) $y_{tt} = c^2 y_{xx}$ for $-\infty < x < \infty$, $t > 0$;

  (2) $y(x, 0) = f(x)$, $y_t(x, 0) = g(x)$ for $-\infty < x < \infty$,

  where initial transverse displacement f and velocity g are given.

- The general solution of (1) is $y(x,t) = F(x-ct) + G(x+ct)$.
- ICs (2) $\implies F(x) + G(x) \underset{(a)}{=} f(x),\ -cF'(x) + cG'(x) = g(x)$ for $x \in \mathbb{R}$.
- The latter implies $-F(x) + G(x) \underset{(b)}{=} \frac{1}{c}\int\limits_0^x g(s)\,\mathrm{d}s + a$ $(a \in \mathbb{R})$.

$$\text{(a)} - \text{(b)} \implies F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int\limits_0^x g(s)\,\mathrm{d}s - \frac{1}{2}a$$

$$\text{(a)} + \text{(b)} \implies F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int\limits_0^x g(s)\,\mathrm{d}s + \frac{1}{2}a$$

- Hence,

$$y(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2c}\int\limits_{x-ct}^0 g(s)\,\mathrm{d}s - \frac{1}{2}a + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int\limits_0^{x+ct} g(s)\,\mathrm{d}s + \frac{1}{2}a$$

$$\implies \boxed{y(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int\limits_{x-ct}^{x+ct} g(s)\,\mathrm{d}s}$$

This is D'Alembert's formula.

Remarks

(1) Don't forget the constant a!

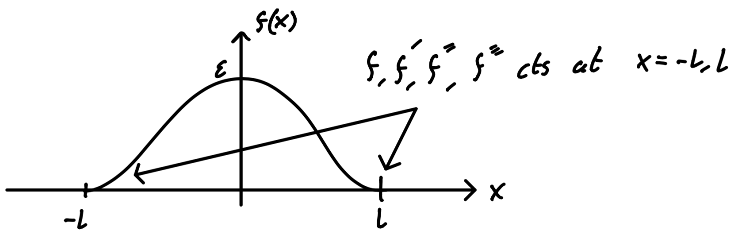
(2) Argument shows $\exists!$ solution of IVP (1)-(2).

(3) Can also prove uniqueness via energy conservation under the additional assumption that $y_t, y_x \to 0$ sufficiently rapidly as $x \to \pm\infty$ that the energy $E(t) = \int\limits_{-\infty}^{\infty} \frac{\rho}{2}y_t^2 + \frac{T}{2}y_x^2\,\mathrm{d}x$ exists.

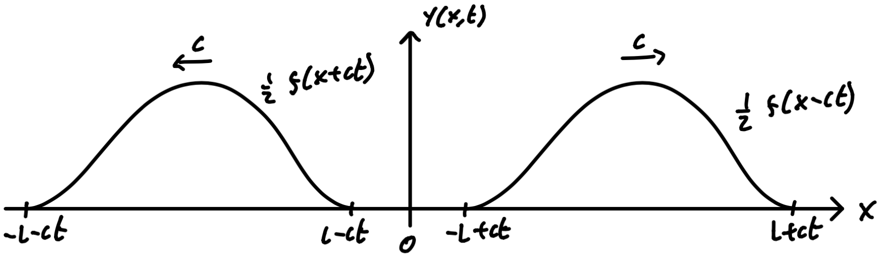
Example 4.5.

$$f(x) = \begin{cases} \epsilon\cos^4\left(\frac{\pi x}{2L}\right) & \text{for } |x| \le L\,, \\ 0 & \text{otherwise,} \end{cases} \qquad g(x) = 0\,,$$

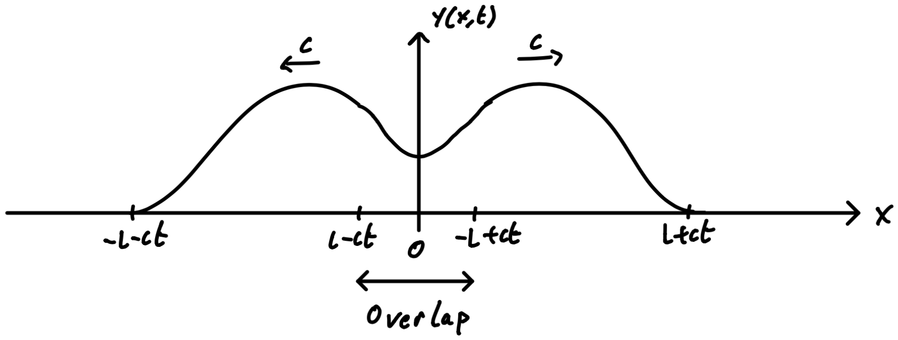
where ϵ, $L \in \mathbb{R}^+$.



DF $\implies y(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct)$.



$\underline{ct > L}$



$0 < ct < L$

Explicit formulae for these graphs requires some careful book keeping - much easier to use a...

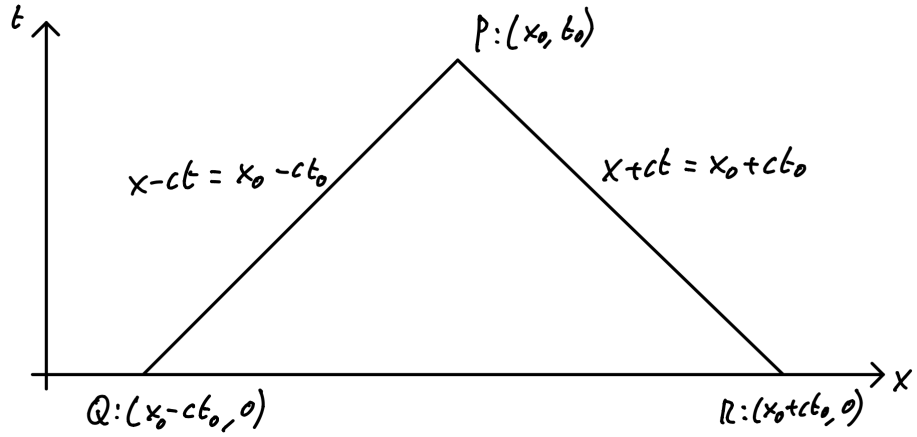
Characteristic diagram

- Let $P = (x_0, t_0) \in \mathbb{R} \times \mathbb{R}^+$. How does $y(P)$ depend on f e.g.?
-

$$\text{DF} \implies y(x_0, t_0) = \frac{1}{2}(f(x_0 - ct_0) + f(x_0 + ct_0)) + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,\mathrm{d}s \qquad (\dagger)$$

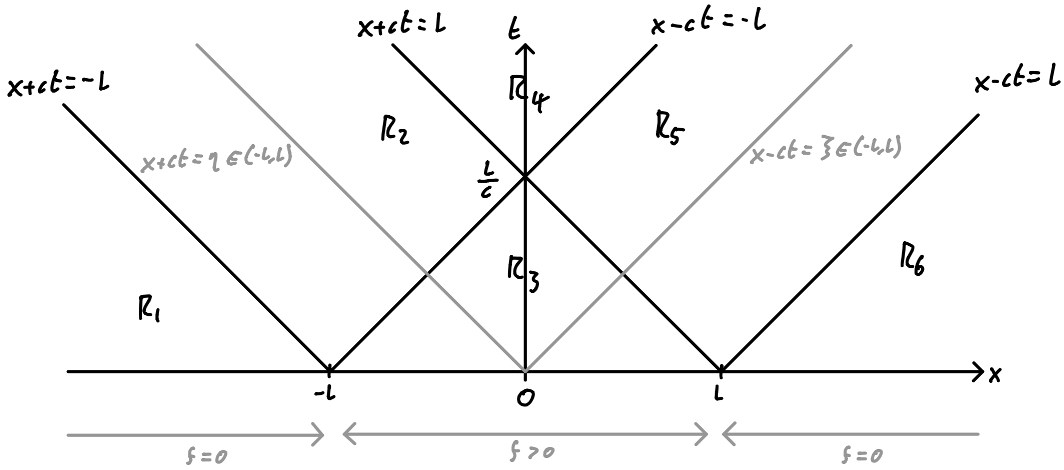
$$\implies y(P) = \frac{1}{2}(f(Q) + f(R)) + \frac{1}{2c}\int_Q^R g(s)\,\mathrm{d}s\,, \qquad (\ddagger)$$

where Q and R are points on the x-axis as shown.



- Note deliberate abuse of notation in (‡) to aid geometric interpretation of (†).
- **Definition:** $x \pm ct = x_0 \pm ct_0$ called characteristic lines through P.
- (‡) $\implies$ $y(P)$ depends only on
  - (i) f through the values f takes at Q and R;
  - (ii) g through the values g takes on x-axis between Q and R.
- **Definition:** The interval $[x_0 - ct_0, x_0 + ct_0]$ of the x-axis between Q and R is called the domain of dependence of $P = (x_0, t_0)$.
- If f or g modified outside the domain of dependence of P, then $y(P)$ is unchanged.
- Exploit geometric interpretation (‡) of DF (†) to construct explicit formulae for the solution: contribution to $y(P)$ from f to g change at points on x-axis where f and g change their analytic behaviour.

- Hence, given a particular f and g, first task is to identify these points on x-axis and sketch the characteristic lines $x \pm c =$ constant through each of them - this is the **characteristic diagram**.
- This divides the (x, t)-plane, with $t > 0$, into regions in which the contributions from f and g may be different.
- Back to the earlier example...
Characteristic diagram



- DF $\implies$ $y(P) = \frac{1}{2}(f(Q) + f(R))$, where P, Q, R are points shown.
- $PQ \parallel x - ct = \pm L$, and $PR \parallel x + ct = \pm L$, so solution as follows:
  - $P \in R_1 \implies y = \frac{1}{2}[0 + 0]$
  - $P \in R_2 \implies y = \frac{1}{2}\left[0 + \epsilon \cos^4\left(\frac{\pi}{2c}(x + ct)\right)\right]$
  - $P \in R_3 \implies y = \frac{1}{2}\left[\epsilon \cos^4\left(\frac{\pi}{2c}(x - ct)\right) + \epsilon \cos^4\left(\frac{\pi}{2c}(x + ct)\right)\right]$
  - $P \in R_4 \implies y = \frac{1}{2}[0 + 0]$
  - $P \in R_5 \implies y = \frac{1}{2}\left[\epsilon \cos^4\left(\frac{\pi}{2c}(x - ct)\right) + 0\right]$
  - $P \in R_6 \implies y = \frac{1}{2}[0 + 0]$
- Since y is continuous on characteristics bounding regions, it does not matter to which region each belongs, e.g. could pick
  - $R_1 : x + ct < -L$, $t > 0$;
  - $R_2 : -L \leq x + ct \leq L$, $x - ct \leq L$;
  - $R_3 : -L < x + ct < L$, $-L < x - ct < L$, $t > 0$;

  etc.

Example 4.6. Suppose $y(x, t)$ such that

(1) $y_{tt} = c^2 y_{xx}$ for $-\infty < x < \infty$, $t > 0$;

(2) $y(x, 0) = f(x)$, $y_t(x, 0) = g(x)$ for $-\infty < x < \infty$.

Find $y(x, t)$ when $f(x) = 0$ and $g(x) = \begin{cases} vx/L & \text{for } |x| \leq L\,, \\ 0 & \text{otherwise,} \end{cases}$, where $L, v \in \mathbb{R}^+$.

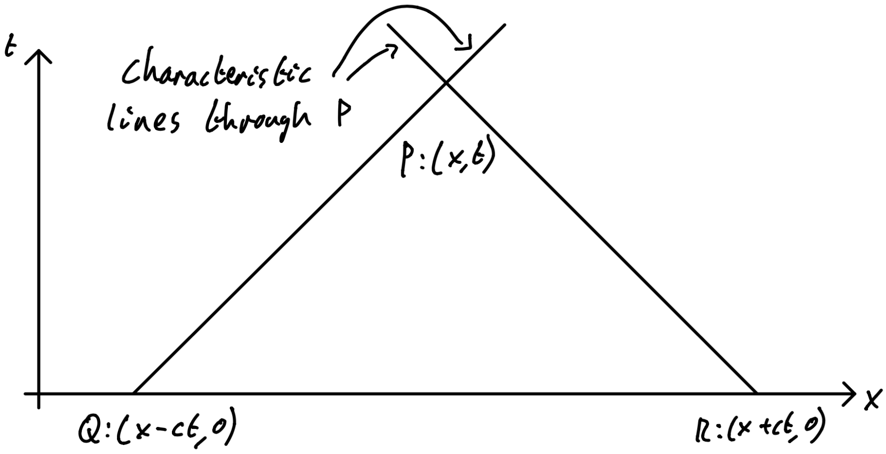
- Recall D'Alembert's Formula (DF) for the solution of (1)-(2):

$$y(x, t) = \frac{1}{2}(f(x - ct) + f(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, \mathrm{d}s$$

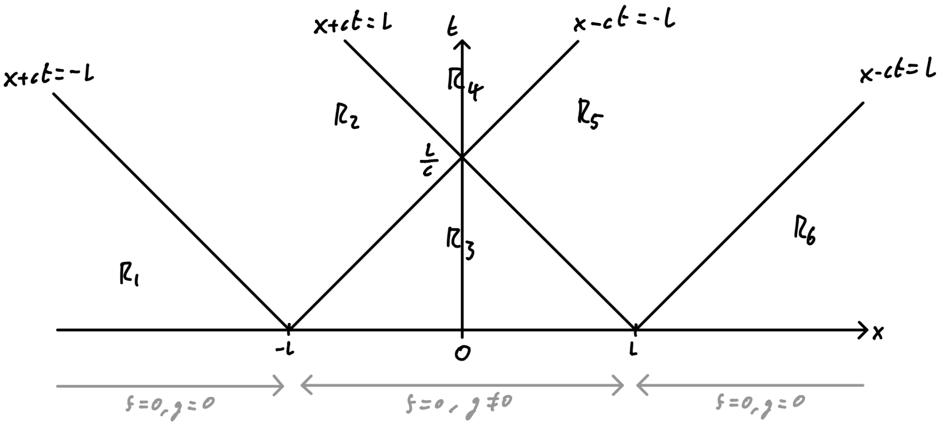
- Thus,

$$y(P) = \frac{1}{2c} \int_{Q}^{R} g(s) \, \mathrm{d}s \,,$$

where P, Q, R are the points shown



Characteristic diagram



- $PQ \parallel x - ct = \pm L$ and $PR \parallel x + ct = \pm L$, so solution as follows:

$$R_1 : y = \frac{1}{2c} \int_{x-ct}^{x+ct} 0 \, \mathrm{d}s = 0$$

$$R_2 : y = \frac{1}{2c} \int_{x-ct}^{-L} 0 \, \mathrm{d}s + \frac{1}{2c} \int_{-L}^{x+ct} \frac{vs}{L} \, \mathrm{d}s = \frac{v}{4Lc} \left((x+ct)^2 - L^2 \right)$$

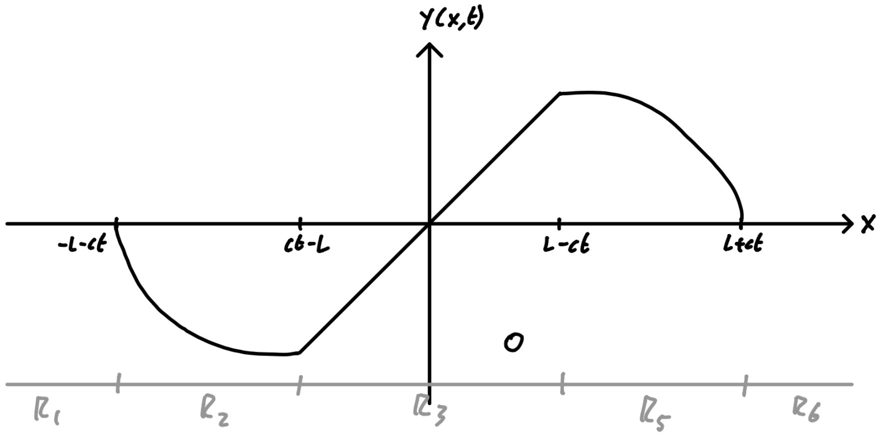
$$R_3 : y = \frac{1}{2c} \int_{x-ct}^{x+ct} \frac{vs}{L} \, \mathrm{d}s = \frac{v}{4Lc} \left((x+ct)^2 - (x-ct)^2 \right) = \frac{vxt}{L}$$

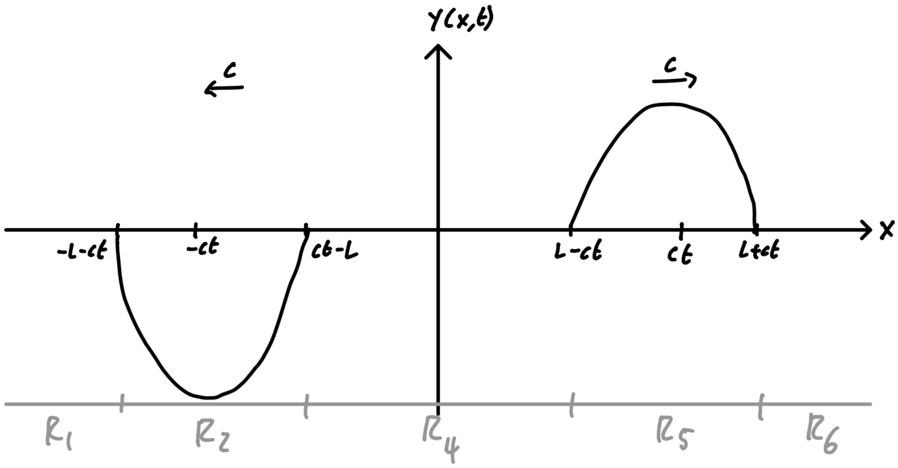
$$R_4 : y = \frac{1}{2c} \int_{x-ct}^{-L} 0 \, \mathrm{d}s + \frac{1}{2c} \int_{-L}^{L} \frac{vs}{L} \, \mathrm{d}s + \frac{1}{2c} \int_{L}^{x+ct} 0 \, \mathrm{d}s = 0$$

$$R_5 : y = \frac{1}{2c} \int_{x-ct}^{L} \frac{vs}{L} \, \mathrm{d}s + \frac{1}{2c} \int_{L}^{x+ct} 0 \, \mathrm{d}s = \frac{v}{4Lc} \left(L^2 - (x-ct)^2 \right)$$

$$R_6 : y = \frac{1}{2c} \int_{x-ct}^{x+ct} 0 \, \mathrm{d}s = 0$$

- Note solution continuous across borders between regions.





- Note there are corners $\implies$ not a classical (twice continuously-differentiable) solution!

Fourier Series & PDEs: Lectures 13-14

Laplace's equation in the plane

- Heat conduction in a rigid isotropic material (e.g. metal) is governed in 3D by the heat equation

$$T_t = \kappa \nabla^2 T \,,$$

where $T(x, y, z, t)$ is the temperature, κ the thermal diffusivity and $\nabla^2 T = T_{xx} + T_{yy} + T_{zz}$.

- Derive in Multivariable Calculus from conservation of energy and Fourier's Law using the Divergence Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \rho C T \,\mathrm{d}V = \iint_{\partial V} \mathbf{q} \cdot (-\mathbf{n}) \,\mathrm{d}S \qquad \text{(Energy)}$$

$$\underset{\text{(Div. Thm)}}{\Longrightarrow} \iiint_V \rho c T_t \,\mathrm{d}V = - \iiint_V \nabla \cdot \mathbf{q} \,\mathrm{d}V$$

$$\Longrightarrow \rho c T_t + \nabla \cdot \mathbf{q} = 0 \,. \qquad \text{(assuming LHS cts)}$$

Substitute $\mathbf{q} \underset{\text{(Fourier's Law)}}{=} -k\nabla T$ to give

$$T_t = \frac{k}{\rho c} \nabla \cdot \nabla T = \kappa \nabla^2 T \,.$$

- In this course we consider 2D steady-state solutions:

$$T = T(x, y) \Longrightarrow \boxed{T_{xx} + T_{yy} = 0} \,.$$

This is **Laplace's equation**.

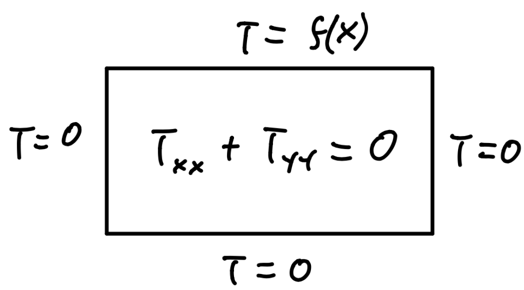
BVP in Cartesian Coordinates

- Find $T(x, y)$ such that

  (1) $T_{xx} + T_{yy} = 0$ for $0 < x < a$, $0 < y < b$;

  (2) $T(0, y) = 0$, $T(a, y) = 0$ for $0 < y < b$;

  (3) $T(x, 0) = 0$, $T(x, b) = f(x)$ for $0 < x < a$;



- Apply Fourier's method.

Step I

- $T = F(x)G(y)$ in (1) $\implies \frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)}$ for $FG \neq 0$.
- LHS independent of y and RHS independent of x $\implies$ LHS=RHS independent of x and y $\implies$ LHS=RHS=$-\lambda \in \mathbb{R}$, say.
- Hence $-F'' = \lambda F$ for $0 < x < a$. (2) and T nontrivial $\implies F(0) = 0$, $F(a) = 0$.
- Solved before! Only nontrivial solutions are $F(x) = B\sin\left(\frac{n\pi x}{a}\right)$, $B \in \mathbb{R}$, for $\lambda = \left(\frac{n\pi}{a}\right)^2$, $n \in \mathbb{N}\setminus\{0\}$.
- $\lambda = \frac{n\pi}{a} \implies G'' - \left(\frac{n\pi}{a}\right)^2 G = 0 \implies G = C\cosh\left(\frac{n\pi y}{a}\right) + D\sinh\left(\frac{n\pi y}{a}\right)$, $C, D \in \mathbb{R}$.
- Combo $\implies$ nontrivial separable solutions of (1)-(2) are

$$T_n(x,y) = \sin\left(\frac{n\pi x}{a}\right)\left(a_n\cosh\left(\frac{n\pi y}{a}\right) + b_n\sinh\left(\frac{n\pi y}{a}\right)\right),$$

  where $a_n, b_n \in \mathbb{R}$ and $n \in \mathbb{N}\setminus\{0\}$.

Step II

Formally superimpose $\implies T(x,y) = \sum\limits_{n=1}^{\infty} T_n(x,y)$ is the general series solution of (1)-(2).

Step III

- BC on $y = 0 \implies 0 = \sum\limits_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{a}\right) \implies a_n = 0 \forall n$.
- BC on $y = b \implies f(x) = \sum\limits_{n=1}^{\infty} b_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$ for $0 < x < a$, so that $b_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int\limits_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x$ by the theory of FS.
- NB: Could also apply BC on $y = 0$ to find $a_n = 0$ at end of Step I, i.e. before superimposing in step III.
- NB: ON sheet consider case in which $a = b = L$ and $f = T^\star \in \mathbb{R}$.

BVP in plane polar coordinates

- In plane polar coordinates (r, θ), Laplace's equation for $T(r,\theta)$ becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \text{ for } r > 0 \qquad (\star)$$

- Start by finding all nontrivial separable solutions that are 2π-periodic in θ.

$$T = F(r)G(\theta) \implies F''G + \frac{1}{r}F'G + \frac{1}{r^2}FG'' = 0$$
$$\underset{\times\frac{r^2}{FG}}{\implies} \frac{r^2F''(r) + rF'(r)}{F(r)} = -\frac{G''(\theta)}{G(\theta)} \qquad (FG \neq 0)$$

- LHS independent of θ, RHS independent of r $\implies$ LHS=RHS independent r and θ $\implies$ LHS=RHS=$\lambda \in \mathbb{R}$.
- Hence, need to find all $\lambda \in \mathbb{R}$ such that $G''(\theta) + \lambda G(\theta) = 0$ has a nontrivial solution $G(\theta)$ of period 2π. Consider cases.

(i) $\underline{\lambda = -\omega^2\ (\omega > 0 \text{ wlog})} \implies G(\theta) = A\cosh\omega\theta + B\sinh\omega\theta,\ (A, B \in \mathbb{R})$.

- G 2π periodic $\implies G(0) = G(\pm 2\pi) \implies A = A\cosh 2\pi\omega \pm B\sinh 2\pi\omega \underset{(+,-)}{\implies} A(\cosh 2\pi\omega - 1) = 0,\ B\sinh 2\pi\omega = 0 \underset{\omega>0}{\implies}$

(ii) $\underline{\lambda = 0} \implies G(\theta) = A + B\theta,\ (A, B \in \mathbb{R})$.

- G 2π-periodic $\implies B = 0$, but A arbitrary admissible.

$$\begin{aligned} r^2F'' + rF' = 0 &\implies r(rF')' = 0 \\ &\implies rF' = d \qquad (r > 0,\ d \in \mathbb{R}) \\ &\implies F = c + d\log r \qquad (c \in \mathbb{R}) \end{aligned}$$

- Combo $\implies T_0 = A_0 + B_0 \log r,\ (A_0, B_0 \in \mathbb{R})$. This is a cylindrically-symmetric solution (i.e. independent of θ).

(iii) $\underline{\lambda = \omega^2\ (\omega > 0 \text{ wlog})} \implies G(\theta) = R\cos(\omega\theta + \Phi),\ (R\Phi \in \mathbb{R})$.

- G nontrivial $\implies R \neq 0 \implies G$ has prime period $\frac{2\pi}{\omega}$.
  Hence, G 2π-periodic and nontrivial $\implies \exists n \in \mathbb{N}\setminus\{0\}$ such that $n\frac{2\pi}{\omega} = 2\pi$, i.e. $\omega = n$ for some $n \in \mathbb{N}\setminus\{0\}$.
  In practice, better to write $G(\theta) = A\cos n\theta + B\sin n\theta$, where $A = R\cos\Phi$, $B = -R\sin\Phi$ are arbitrary real constants.
- $\lambda = n^2 \implies r^2F'' + rF' - n^2F = 0$ for $r > 0$ (Euler's ODE).
  Let $r = e^t$,, $F(r) = W(t)$, then $\frac{\mathrm{d}W}{\mathrm{d}t} = \frac{\mathrm{d}F}{\mathrm{d}r}\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}F}{\mathrm{d}r}$, so

$$\frac{\mathrm{d}^2W}{\mathrm{d}t^2} = \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right)\frac{\mathrm{d}r}{\mathrm{d}t} = r\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}F}{\mathrm{d}r}\right) = r^2F'' + rF' = n^2F = n^2W\,.$$

  $W = e^{\mu t} \implies \mu^2 = n^2 \implies W$ has general solution $W = Ce^{nt} + De^{-nt}$, $(C, D \in \mathbb{R}) \implies F$ has general solution

$$F(r) = Cr^n + Dr^{-n} \qquad (C, D \in \mathbb{R})$$

  NB: Alternatively, let $F(r) = r^\mu$, then $\mu(\mu-1) + \mu - \mu^2 = 0 \implies \mu^2 = n^2 \implies \mu = \pm n \implies$ general solution as above by theory of linear 2nd order ODEs.
- Combo $\implies T_n = (A_n r^b + B_n r^{-n})\cos n\theta + (C_n r^n + D_n r^{-n})\sin n\theta$, where $A_n = AC$, $B_n = AD$, $C_n = BC$, $D_n = BD$ are arbitrary real constants and $n \in \mathbb{N}\setminus\{0\}$.
- Superimpose $\implies$ general series solution of $(\star)$ is

$$T(r,\theta) = \sum_{n=0}^{\infty} T_n(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\Big((A_n r^n + B_n r^{-n})\cos n\theta + (C_n r^n + D_n r^{-n})\sin n\theta\Big)\,.$$

BVP in plane polar coordinates continued

- Last lecture we showed that the general series solution of

$$\frac{\mathrm{d}^2T}{\mathrm{d}r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial\theta^2} = 0 \qquad (r > 0)$$

  is given by

$$T(r,\theta) = \sum_{n=0}^{\infty} T_n(r,\theta) = A_0 + B_0\log r + \sum_{n=1}^{\infty}\Big((A_n r^n + B_n r^{-n})\cos n\theta + (C_n r^n + D_n r^{-n})\sin n\theta\Big)\,, \quad (\star)$$

  where $A_n, B_n, C_n, D_n \in \mathbb{R}$.

Example 5.1. Find T such that

(1) $\nabla^2 T = 0$ in $a < r < b$,

(2) $T = T_0^\star$ on $r = a$, $T = T_1^\star$ on $r = b$,

where $a, b, T_0^\star, T_1^\star \in \mathbb{R}$.

- (1) $\implies$ ($\star$) pertains. BCs (2) can only be satisfied if

$$T_0^\star = A_0 + B_0 \log a + \sum_{n=1}^{\infty} \left((A_n a^n + B_n a^{-n}) \cos n\theta + (C_n a^n + D_n a^{-n}) \sin n\theta \right) ,$$

$$T_1^\star = A_0 + B_0 \log b + \sum_{n=1}^{\infty} \left((A_n b^n + B_n b^{-n}) \cos n\theta + (C_n b^n + D_n b^{-n}) \sin n\theta \right) ,$$

each for $-\pi < \theta \leq \pi$, say.

- Since the Fourier coefficients of a Fourier series are unique, we can equation them on LHS and RHS of each equality $\implies$

$$\begin{bmatrix} 1 & \log a \\ 1 & \log b \end{bmatrix} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix} , \underbrace{\begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \begin{bmatrix} a^n & a^{-n} \\ b^n & b^{-n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{n \in \mathbb{N} \setminus \{0\}}$$

$$\underset{a<b}{\implies} \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} = \frac{1}{\log\left(\frac{b}{a}\right)} \begin{bmatrix} \log b & -\log a \\ -1 & 1 \end{bmatrix} \begin{bmatrix} T_0^\star \\ T_1^\star \end{bmatrix} , \underbrace{\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{n \in \mathbb{N} \setminus \{0\}}$$

$$\implies T = \frac{T_0^\star \log b - T_1^\star \log a}{\log\left(\frac{b}{a}\right)} + \frac{T_1^\star - T_0^\star}{\log\left(\frac{b}{a}\right)} \log r .$$

Dimensionally correct?

$$T = T_0^\star \frac{\log\left(\frac{r}{b}\right)}{\log\left(\frac{a}{b}\right)} + T_1^\star \frac{\log\left(\frac{r}{b}\right)}{\log\left(\frac{b}{a}\right)} \quad \checkmark$$

- NB: Alternatively, we could have sought a circularly-symmetric solution $T = T(r)$ from the outset because boundary data is independent of θ. However, method above generalises to $T_0^\star$ and $T_1^\star$ being functions of θ.

Example 5.2. Find T such that

(1) $\nabla^2 T = 0$ in $r < a$,

(2) $T(a, 0) = T^\star \sin^3 \theta$ for $-\pi < \theta \leq \pi$,

where $a, T^\star \in \mathbb{R}^+$.

(1) $\implies$ T must be twice differentiable with respect to x and y at origin
$\implies$ T must certainly be continuous and therefore bounded at origin
$\implies$ ($\star$) pertains but with $B_n = 0 \; \forall n \in \mathbb{N}$ and $D_n = 0 \; \forall n \in \mathbb{N} \setminus \{0\}$.

- (2) then requires

$$T^\star \sin^3 \theta = A_n + \sum_{n=1}^{\infty} (A_n a^n \cos n\theta + B_n a^n \sin n\theta)$$

for $-\pi < \theta \leq \pi$.
But the FS for the LHS is given by the identity

$$T^\star \sin^3 \theta = \frac{3T^\star}{4} \sin \theta - \frac{T^\star}{4} \sin 3\theta$$

so equating Fourier coefficients gives

$$B_1 a = \frac{3T^\star}{4},\ B_3 a^3 = -\frac{T^\star}{4}$$

and the rest vanish.
Hence, $T = \frac{3T^\star}{4}\left(\frac{r}{a}\right)\sin\theta - \frac{T^\star}{4}\left(\frac{r}{a}\right)^3 \sin 3\theta$.

- Question: What is the heat flux out of the disc through $r = a$?
- Answer: Outward pointing unit normal $\mathbf{n} = \mathbf{e}_r$, so by Fourier's Law

$$\mathbf{q}\cdot\mathbf{n}|_{r=a} = (-k\nabla T)\cdot\mathbf{e}_r|_{r=a} = -kT_r(a,\theta) = -k\left(\frac{3T^\star}{4a}\sin\theta - \frac{3T^\star}{4a}\sin 3\theta\right).$$

- NB: $\nabla^2 T = 0 \iff \nabla\cdot\mathbf{q} = 0 \implies \int\limits_{r=a}\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}s = \iint\limits_{r<a}\nabla\cdot\mathbf{q}\,\mathrm{d}x\,\mathrm{d}y = 0$, so zero net flux through $r = a$ as there's no volumetric heating.

Poisson's Integral Formula

- Find T such that $\nabla^2 T = 0$ in $r < a$ with $T(a,\theta) = f(\theta)$ for $-\pi < \theta \le \pi$, where $a \in \mathbb{R}^+$ and f is given.
- As in the last example, general series solution is given by $(\star)$ with $B_n = 0\ \forall n \in \mathbb{N}$, $D_n = 0\ \forall n \in \mathbb{N}\backslash\{0\}$, so BC satisfied if

$$f(\theta) = \underbrace{A_0}_{\frac{a}{2}} + \sum_{n=1}^{\infty}\left(\underbrace{A_n a^n}_{a_n}\cos n\theta + \underbrace{B_n a^n}_{b_n}\sin n\theta\right)\quad\text{for } -\pi < \theta \le \pi\,.$$

- Theory of FS then gives

$$A_0 = \frac{a}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,\mathrm{d}\phi$$

$$A_n = \frac{a_n}{a^n} = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\cos(n\phi)\,\mathrm{d}\phi \qquad (n \in \mathbb{N}\setminus\{0\})$$

$$B_n = \frac{b_n}{a^n} = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\phi)\sin(n\phi)\,\mathrm{d}\phi\,, \qquad (n \in \mathbb{N}\setminus\{0\})$$

where we have introduced a dummy variable ϕ for convenience below.

- Given a particular f, can evaluate these expressions (see example 5.2), but remarkably we can sum for general f (sufficiently).
- Substitute Fourier coefficients into general series solution and assume $\sum\int = \int\sum$ gives

$$\begin{aligned}T(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,\mathrm{d}\phi + \sum_{n=1}^{\infty}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{r}{a}\right)^n[\cos(n\theta)\cos(n\phi) + \sin(n\theta)\sin(n\phi)]f(\phi)\,\mathrm{d}\phi\right)\\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\phi)\right)f(\phi)\,\mathrm{d}\phi\,.\end{aligned}$$

- Let $\alpha = \theta - \phi$ and $z = \frac{r}{a}e^{i\alpha}$, then

$$\begin{aligned}
\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos n\alpha &= \Re\left(\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n e^{in\alpha}\right) \\
&= \Re\left(\frac{1}{2} + \sum_{n=1}^{\infty} z^n\right) \\
&= \Re\left(\frac{1}{2} + \frac{z}{1-z}\right) && (|z| < 1) \\
&= \frac{1}{2}\Re\left(\frac{1+z}{1-z}\right) \\
&= \frac{a^2 - r^2}{2(a^2 - 2ar\cos\alpha + r^2)} \, . && (a = \tfrac{a}{r}e^{i\alpha})
\end{aligned}$$

- Hence, obtain PIF:

$$T(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{f(\phi)\,\mathrm{d}\phi}{a^2 - 2ar\cos(\theta - \phi) + r^2} \, . \qquad (r < a)$$

- NB: $r = 0 \implies T = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\,\mathrm{d}\phi$.
  This means temperature at the centre of the disc is the average of the temperature profile around the boundary.

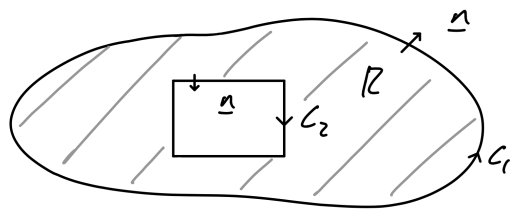
- Next time: uniqueness.

Fourier Series & PDEs: Lectures 15-16

Uniqueness

Green's Theorem in the plane (Divergence Theorem Form)

Let R be a closed bounded region in the (x, y)-plane, whose boundary ∂R is the union $C_1 \cup C_2 \cup \cdots \cup C_m$ of a finite number of piecewise smooth simple closed curves.



Let $\mathbf{F} = (F_1(x, y), F_2(x, y))$ be continuous and have continuous first order derivatives on $R \cup \partial R$. Then

$$\iint_R \nabla \cdot \mathbf{F} \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} \mathbf{F} \cdot \mathbf{n} \,\mathrm{d}s\,,$$

where $\mathbf{n}$ is the outward pointing unit normal to ∂R in the (x, y)-plane and $\mathrm{d}s$ an element of arclength.

Example 5.3. Derivation of the 2D inhomogeneous heat equation

$$[\text{Energy}]: \quad \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\iint_R \rho c T \,\mathrm{d}x\,\mathrm{d}y}_{\text{Rate of change of internal heat energy}} = \underbrace{\int_{\partial R} \mathbf{q} \cdot (-\mathbf{n}) \,\mathrm{d}s}_{\text{Net heat flux into } R \text{ through } \partial R} + \underbrace{\iint_R Q \,\mathrm{d}x\,\mathrm{d}y}_{Volumetricheating}$$

- NB: [each term]=$\mathrm{J\,m^{-1}\,s^{-1}}$ since this is per unit distance in the z-direction.
- Assuming T_t continuous on $R \cup \partial R$ and using Green's Theorem with $\mathbf{F} = \mathbf{q}$ gives

$$\iint_R \rho c T_t + \nabla \cdot \mathbf{q} - Q \,\mathrm{d}x\,\mathrm{d}y = 0\,.$$

- Assuming integrand continuous, R arbitrary $\implies \rho c T_t + \nabla \cdot \mathbf{q} = Q$.
- Finally, Fourier's Law $q = -k\nabla T$ gives $\rho c T_t = \nabla \cdot (k \nabla T) + Q$.

Uniqueness for the Dirichlet problem

Theorem 5.1. Suppose $T(x, y)$ such that $\nabla^2 T = 0$ in R with $T = f$ on ∂R (Dirichlet problem), where R as in Green's Theorem and path-connected and f given. Then the BVP has at most one solution.

Proof. Let W be the difference between two solutions, then linearity gives

(1) $\nabla^2 W = 0$ in R,

(2) $W = 0$ on ∂R.

Trick: let $\mathbf{F} = W\nabla W = \nabla\left(\frac{1}{2}W^2\right)$ in Green's Theorem.

- Then $\iint_R \nabla \cdot W \nabla W \,\mathrm{d}x\,\mathrm{d}y = \int_{\partial R} W \nabla W \cdot \mathbf{n} \,\mathrm{d}s$.

- But

$$\begin{aligned}(1) &\implies \nabla\cdot(W\nabla W) = W\nabla^2 W + \nabla W\cdot\nabla W = |W|^2 \text{ in } R\\ (2) &\implies W\nabla W\cdot\mathbf{n} = 0 \text{ on } \partial R\,,\end{aligned}$$

so $\iint\limits_R |\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y = 0$.

- Assuming ∇W is continuous on $R\cup\partial R$, this implies $\nabla W = \mathbf{0}$ on $R \implies W =$ constant on R (as it's path connected).
- But $W = 0$ on ∂R, so assuming W is continuous on $R\cup\partial R$, the constant must vanish, so that $W = 0$ on $R\cup\partial R$.

□

Example 5.4. Find T such that $\nabla^2 T = 0$ in $r < a$ with $T = T^\star\frac{x}{a}$ on $r = a$.

- If we can find any solution, then the uniqueness theorem guarantees it is the only solution.
- Could proceed via general series solution or Poisson's Integral Formula, but quicker to spot $T = T^\star\frac{x}{a}$.

Example 5.5. Find T such that $\nabla^2 T = 0$ in $r > a$ with $T = T^\star\frac{x}{a}$ on $r = a$ and T bounded as $r\to\infty$.

- Spot $B_1 r^{-1}\cos\theta$ is a solution provided $B_1 a^{-1} = T^\star$.
- Question: is it the only solution?
- Answer: Uniqueness theorem above not applicable because R not bounded. But, if W is the difference between two solutions, then for fixed $b > a$

$$\iint\limits_{a<r<b} |\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y \underset{\nabla^2 W=0 \text{ in } a<r<b}{=} \iint\limits_{a<r<b} \nabla\cdot(W\nabla W)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{r=b} \underbrace{W\nabla W\cdot\mathbf{e}_r\,\mathrm{d}s}_{W\frac{\partial W}{\partial r} r\,\mathrm{d}\theta} - \underbrace{\int\limits_{r=b} W\nabla W\cdot\mathbf{e}_r\,\mathrm{d}s}_{=0 \text{ since } W=0 \text{ on } r=a}\,.$$

So we have uniqueness provided $rW\frac{\partial W}{\partial r}\to 0$ as $r\to\infty$, which is the case if e.g. $r\frac{\partial T}{\partial r}\to 0$ as $r\to\infty$.

Uniqueness for the Neumann Problem

Theorem 5.2. Suppose $T(x,y)$ such that $\nabla^2 T = F$ in R with $\frac{\partial T}{\partial n}\equiv \mathbf{n}\cdot\nabla T = g$ on ∂R (Neumann problem), where R as in Green's Theorem and path-connected and F, g given. Then the BVP has no solution unless

$$\iint\limits_R F\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial R} g\,\mathrm{d}s\,.$$

When a solution exists, it is not unique: any two solutions differ by a constant.

Proof.

- Suppose there is a solution T and let $\mathbf{F} = \nabla T$ in Green's Theorem, then

$$\iint\limits_R F\,\mathrm{d}x\,\mathrm{d}y = \iint\limits_R \nabla\cdot(\nabla T)\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\partial R} \nabla T\cdot\mathbf{n}\,\mathrm{d}s = \int\limits_{\partial R} g\,\mathrm{d}s\,.$$

- Now let W be the difference between two solutions, so that $\nabla^2 W = 0$ in R and $\frac{\partial W}{\partial n} = 0$ on ∂R. Then

$$\iint\limits_R |\nabla W|^2\,\mathrm{d}x\,\mathrm{d}y \underset{\text{PDE}}{=} \iint\limits_R \nabla\cdot(W\nabla W)\,\mathrm{d}x\,\mathrm{d}y \underset{\text{Green's Theorem}}{=} \int\limits_{\partial R} W\nabla W\cdot\mathbf{n}\,\mathrm{d}s \underset{\text{BC}}{=} 0\,,$$

as before.

- Assuming ∇W is continuous on $R \cup \partial R$, this imples $\nabla W = \mathbf{0}$ on R, so that $W =$ constant on R. Hence, $W =$ constant on $R \cup \partial R$ assuming W is continuous there.

□

Example 5.6. Find T such that $\nabla^2 T = 0$ in $r < a$ with $\frac{\partial T}{\partial n} = g(\theta)$ on $r = a$, where g is given.

- General series solution of $\nabla^2 T = 0$ in $r < a$ is

$$T = A_0 + \sum_{n=1}^{\infty} \left(A_n r^n \cos n\theta + B_n r^n \sin n\theta \right) .$$

- On $r = a$, $\frac{\partial T}{\partial n} = \mathbf{n} \cdot \nabla T = \frac{\partial T}{\partial r}$, so BC can be satisfied only if

$$g(\theta) = \sum_{n=1}^{\infty} \left(n A_n a^{n-1} \cos n\theta + n B_n a^{n-1} \sin n\theta \right) \text{ for } -\pi < \theta \leq \pi \qquad \text{(say)}$$

- Theory of FS then requires

$$0 = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \, \mathrm{d}\theta \qquad (\dagger)$$

$$n A_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \cos n\theta \, \mathrm{d}\theta \qquad (n \in \mathbb{N} \setminus \{0\})$$

$$n B_n a^{n-1} = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\theta) \sin n\theta \, \mathrm{d}\theta \, . \qquad (n \in \mathbb{N} \setminus \{0\})$$

- Two cases: Either
  (i) g such that (†) not true, in which case there is no solution; or
  (ii) g such that (†) is true, in which case the solution is not unique (since A_0 is arbitrary, while rest of Fourier coefficients are uniquely determined).

- This agrees with uniqueness theorem, which guarantees that in case (II) we've found all possible solutions.

Well-posedness

- An IBVP or BVP is wellposed if ∃! solution that depends continuously on the data in the ICs and/or BCs.

Example 6.1. Wave equation

- Suppose $y(x, t)$ such that
  (i) $y_{tt} = y_{xx}$ for $-\infty < x < \infty$, $t > 0$,
  (ii) $y(x, 0) = f(x)$, $y_t(x, 0) = g(x)$ for $-\infty < x < \infty$,

  where f, g are given.

- By D'Alembert's Formula ∃! solution since (1)-(2) $\implies$

$$y(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2} \int_{x-t}^{x+t} g(s) \, \mathrm{d}s \, .$$

- Now change data f, g to F, G, and let Y be the new solution:

$$Y(x,t) = \frac{1}{2}(F(x-ct) + F(x+ct)) + \frac{1}{2}\int_{x-t}^{x+t} G(s)\,\mathrm{d}s\,.$$

- Suppose $\exists \delta > 0$ such that $|f(x) - F(x)| < \delta$, $|g(x) - G(x)| < \delta$ $\forall x \in \mathbb{R}$. (†)
  Then

$$\begin{aligned}
|y(x,t) - Y(x,t)| &= \left|\frac{1}{2}(f(x-t) - F(x-t)) + \frac{1}{2}(f(x+t) - F(x+t))\right| + \frac{1}{2}\int_{x-t}^{x+t} g(s) - G(s)\,\mathrm{d}s \\
&\leq \frac{1}{2}\left|f(x-t) - F(x-t)\right| + \frac{1}{2}\left|f(x+t) - F(x+t)\right| + \frac{1}{2}\int_{x-t}^{x+t} |g(s) - G(s)|\,\mathrm{d}s \\
&\leq \frac{1}{2}\delta + \frac{1}{2}\delta + \frac{1}{2}\cdot 2t \cdot \delta \\
&= (1+t)\delta \text{ for } -\infty < x < \infty,\ t \geq 0\,. \qquad (‡)
\end{aligned}$$

- Fix any $T > 0$ and any $\epsilon > 0$. If we pick $\delta = \frac{\epsilon}{1+T}$ in (†), then (‡) implies

$$|y(x,t) - Y(x,t)| \leq \epsilon\frac{1+t}{1+T} < \epsilon \text{ for } -\infty < x < \infty,\ 0 < t < T\,.$$

- In this sense, the solution depends continuously on the data and the IVP is well-posed.

Example 6.2. Try IVP for Laplace's equation!

- Suppose $y(x,t)$ such that
  (i) $y_{xx} + y_{tt} = 0$ for $-\infty < x < \infty$, $t > 0$,
  (ii) $y(x,0) = f(x)$, $y_t(x,0) = g(x)$ for $-\infty < x < \infty$,

  where f, g are given.
- Problem (I): $f_1 = 0$, $g_1 = 0 \implies y_1 = 0$ is a solution.
- Problem (II): $f_2 = 0$, $g_2 = \delta\cos\left(\frac{x}{\delta}\right) \implies y_2 = \delta^2\cos\left(\frac{x}{\delta}\right)\sinh\left(\frac{t}{\delta}\right)$ is a solution for any $\delta > 0$.
- Observe that $|f_1(x) - f_2(x)| = 0$, $|g_1(x) - g_2(x)| \leq \delta$ $\forall x \in \mathbb{R}$.
- But $|y_1(0,t) - y_2(0,t)| = \delta^2\sinh\left(\frac{t}{\delta}\right) \to \infty$ as $\delta \to 0^+$ for any fixed $t > 0$, so cannot make $|y_1(0,t) - y_2(0,t)| < \epsilon$ for all $0 < t < T$ by taking δ suitably small.
- IVP for Laplace's equation is not well-posed - called ill-posed!

Summary

1. Introduced theory of Fourier Series
   - Periodic, even and odd functions and periodic extensions;
   - Euler's formulae for Fourier coeffs via orthogonality relations;
   - Statement of a powerful pointwise convergence theorem;
   - Related rate of convergence to smoothness;
   - Discussed Gibb's phenomenon - try to avoid!
2. Heat equation

- Derivation in 1D, 2D and 3D;
- Simple solutions;
- Units and nondimensionalisation;
- Fourier's method for IBVPs;
- Generalised to inhomogeneous heat equation and BCs.
- Uniqueness

3. <u>Wave equation</u>
    - Derivation in 1D with gravity and air resistance;
    - Normal modes and natural frequencies;
    - Fourier's method for IBVPs - plucked and flicked strings;
    - Forced and dampled wave equation with inhomogeneous BCs;
    - Normal modes for composite and weighted strings;
    - D'Alembert's solution and characteristic diagrams;
    - Uniqueness.
4. <u>Laplace's equation</u>
    - Derivation in 2D and 3D;
    - Fourier's method for BVPs in (x, y) and (r, θ);
    - Poisson's Integral Formula for Dirichlet problem on a disk;
    - Uniqueness of Dirichlet problem;
    - Nonexistence and nonuniqueness of Neumann problem;
5. <u>Well-posedness</u>
    - Introduced concepts developed later on in course.

<u>Final comments</u>

- Problem sheet questions not too far from prelims questions, so should set you up well for the exam.
- Should try at least 3-5 past papers, but maybe avoid the TT 2015 paper - it turned out much tougher than anticipated.