Fourier series for functions of period 2π

- Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function of period 2π. We want an expansion for f of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)). \qquad (\star)$$

- Q1: If $(\star)$ is true, can we find the constants a_n, b_n in terms of f?
- Q2: With these a_n and b_n, when is $(\star)$ true?

Question 1

- Suppose $(\star)$ is true and that we can integrate it term by term, then

$$\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \mathrm{d}x + \sum_{n=1}^{\infty} \Big(a_n \underbrace{\int_{-\pi}^{\pi} \cos(nx)\,\mathrm{d}x}_{0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin(nx)\,\mathrm{d}x}_{0} \Big),$$

giving

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x\,,$$

i.e. $a_0/2$ is the mean of f over a period.

- **Lemma:** Let $m, n \in \mathbb{N} \setminus \{0\}$. Then we have the orthogonality relations:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \pi\delta_{mn}, \quad \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x = 0 \quad \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \pi\delta_{mn},$$

where δ_{mn} is Kronecker's delta. The proof is on the first problem sheet.

- Fix $m \in \mathbb{N} \setminus \{0\}$, multiply $(\star)$ by $\cos(mx)$ and assume that the orders of summation and integration may be interchanged:

$$\begin{aligned}
\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x &= \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos(mx)\,\mathrm{d}x \\
&\quad + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x + b_n \int_{-\pi}^{\pi} \cos(mx)\sin(nx)\,\mathrm{d}x \right) \\
&= \frac{1}{2}a_0 \cdot 0 + \sum_{n=1}^{\infty} (a_n \pi \delta_{mn} + b_n \cdot 0) \\
&= \pi a_m\,,
\end{aligned}$$

giving

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x \quad \text{for} \quad m \in \mathbb{N} \setminus \{0\}.$$

- Similarly, fix $m \in \mathbb{N} \setminus \{0\}$, multiply $(\star)$ by $\sin(mx)$ and assume that the orders of summation and integration may be interchanged to obtain

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x \quad \text{for} \quad m \in \mathbb{N} \setminus \{0\}.$$

- **Definition:** Suppose f is 2π-periodic and such that the Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad (n \in \mathbb{N}), \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad (n \in \mathbb{N}\setminus\{0\})$$

exist. Then we write

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big),$$

where $\sim$ means the RHS is the Fourier series for f, regardless of whether or not it converges to f.