Convergence of Fourier series

- **Definition:** The right-hand limit of f at c is $f(c_+) = \lim_{\substack{h\to 0\\ h>0}} f(c+h)$ if it exists.

- **Definition:** The left-hand limit of f at c is $f(c_-) = \lim_{\substack{h\to 0\\ h<0}} f(c+h)$ if it exists.

- **Remarks:**

  (1) $f(c)$ need not be defined for $f(c_+)$ or $f(c_-)$ to exist.

  (2) Existence part is important, *e.g.* if $f(x) = \sin(1/x)$ for $x \neq 0$, then $f(0_\pm)$ do not exist.

  (3) f is continuous at c if and only if $f(c_-) = f(c) = f(c_+)$.

  (4) In Example 2, f is continuous for $x \neq k\pi,\ k \in \mathbb{Z}$ with *e.g.* $f(0_+) = 1,\ f(0_-) = -1,\ f(\pi_+) = -1,\ f(\pi_-) = 1$.

- **Definition:** f is piecewise continuous on $(a,b) \subseteq \mathbb{R}$ if there exists a finite number of points $x_1, \ldots, x_m \in \mathbb{R}$ with $a = x_1 < x_2 < \ldots < x_m = b$ such that

  (i) f is defined and continuous on (x_k, x_{k+1}) for all $k = 1, \ldots, m-1$;

  (ii) $f(x_{k+})$ exists for $k = 1, \ldots, m-1$;

  (iii) $f(x_{k-})$ exists for $k = 2, \ldots, m$.

- **Remarks:**

  (1) Note that f need not be defined at its exceptional points $x_1, \ldots, x_m$!

  (2) The functions in Examples 1 and 2 are piecewise continuous on any interval $(a,b) \subseteq \mathbb{R}$.

- **Fourier Convergence Theorem (FCT):** Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic, with both f and f' piecewise continuous on $(-\pi, \pi)$. Then, the Fourier coefficients a_n and b_n exist, and

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for} \quad x \in \mathbb{R}.$$

- **Remarks:**

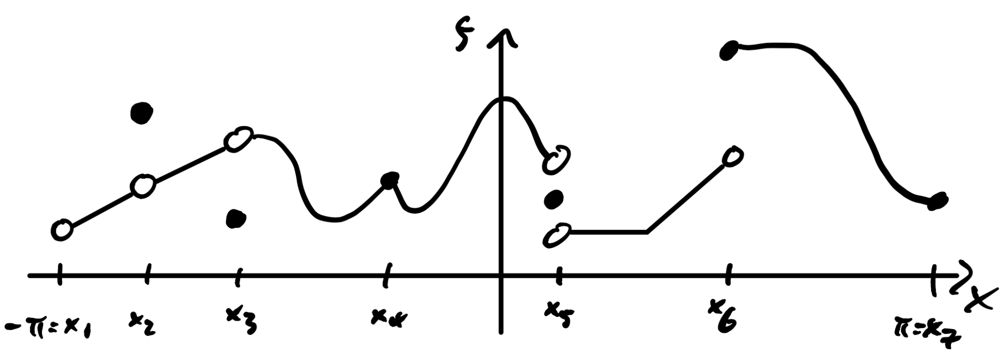
  (1) If f and f' are piecewise continuous on $(-\pi, \pi)$, then there exist $x_1, \ldots, x_m \in \mathbb{R}$ with $-\pi = x_1 < x_2 < \ldots < x_m = \pi$ such that

  (i) f and f' are continuous on (x_k, x_{k+1}) for $k = 1, \ldots, m-1$.

  (ii) $f(x_{k+})$ and $f'(x_{k+})$ exist for $k = 1, \ldots, m-1$.

  (iii) $f(x_{k-})$ and $f'(x_{k-})$ exist for $k = 2, \ldots, m$.

  Thus, in any period f, f' are continuous except possibly at a finite number of points. At each such point f' need not be defined, and one or both of f and f' may have a jump discontinuity, as illustrated for the various different possibilities in the schematic below



For example, if

$$f(x) = \begin{cases} x^{1/2} & \text{for } 0 \le x \le \pi\,, \\ 0 & \text{for } -\pi < x < 0, \end{cases} \qquad \text{then} \quad f'(x) = \begin{cases} \frac{1}{2}x^{-1/2} & \text{for } 0 < x < \pi\,, \\ 0 & \text{for } -\pi < x < 0\,, \\ \text{undefined} & \text{for } x = 0, \pi. \end{cases}$$

Hence, while f is piecewise continuous on $(-\pi, \pi)$, f' is not because $f'(0_+)$ does not exist.

(2) The partial sums of the Fourier series are defined by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \big(a_n \cos(nx) + b_n \sin(nx)\big) \quad \text{for} \quad N \in \mathbb{N}\backslash\{0\}.$$

The theorem states that the partial sums converge pointwise in the sense that

$$\lim_{N\to\infty} S_N(x) = \frac{1}{2}\big(f(x_+) + f(x_-)\big) \quad \text{for each} \quad x \in \mathbb{R}.$$

If f has a jump discontinuity at x so that $f(x_+) \neq f(x_-)$, then the Fourier series converges to the average of the left- and right-hand limits of f at x. If f is continuous at x so that $f(x_+) = f(x_-) = f(x)$, then the Fourier series converges to $\big(f(x_+) + f(x_-)\big)/2 = f(x)$. If f is defined only on *e.g.* $(-\pi, \pi]$, then the FCT holds for its 2π-periodic extension.

(3) While the proof is not examinable, it is amenable to methods from Prelims Analysis as follows:

(i) use the integral expressions for the Fourier coefficients and properties of periodic, even and odd functions to manipulate the partial sums into the form

$$S_N(x) - \frac{1}{2}\big(f(x_+) + f(x_-)\big) = \int_0^{\pi} F(x,t) \sin\left[\left(N + \frac{1}{2}\right)t\right] \mathrm{d}t,$$

where

$$F(x,t) = \frac{1}{\pi}\left(\frac{f(x+t) - f(x_+)}{t} + \frac{f(x-t) - f(x_-)}{t}\right)\left(\frac{t}{2\sin(t/2)}\right);$$

(ii) use the Mean Value Theorem (of Analysis II) to show that $F(x,t)$ is a piecewise continuous function of t on $(0, \pi)$, and hence deduce from the Riemann-Lebesgue Lemma (of Analysis III) that

$$\int_0^{\pi} F(x,t) \sin\left[\left(N + \frac{1}{2}\right)t\right] \mathrm{d}t \to 0 \qquad \text{as } N \to \infty\,.$$

(4) The Fourier series can be integrated termwise under weaker conditions, e.g. if f is only 2π-periodic and piecewise continuous on $(-\pi, \pi)$, then the FCT implies

$$\int_0^{x} f(x)\,\mathrm{d}x = \frac{1}{2}a_0 x + \sum_{n=1}^{\infty}\left(a_n \int_0^{x} \cos(nx)\,\mathrm{d}x + b_n \int_0^{x} \sin(nx)\,\mathrm{d}x\right) \quad \text{for} \quad x \in \mathbb{R}.$$

Note that the integral on the LHS is 2π-periodic if and only if $a_0 = 0$. However, we need stronger conditions to differentiate termwise, *e.g.* if f is 2π-periodic and continuous on $\mathbb{R}$ with both f' and f'' piecewise continuous on $(-\pi, \pi)$, then the FTC implies

$$\frac{1}{2}\big(f'(x_+) + f'(x_-)\big) = \sum_{n=1}^{\infty}\left(a_n \frac{\mathrm{d}}{\mathrm{d}x}(\cos(nx)) + b_n \frac{\mathrm{d}}{\mathrm{d}x}(\sin(nx))\right) \quad \text{for} \quad x \in \mathbb{R}.$$