Rate of convergence

- The smoother f, *i.e.* the more continuous derivatives it has, the faster the convergence of the Fourier series for f.
- If the first jump discontinuity is in the p^{th} derivative of f, with the convention that $p = 0$ if there is a jump discontinuity in f, then typically the slowest decaying a_n and b_n decay like $1/n^{p+1}$ as $n \to \infty$.
- This is an extremely useful result in practice (*e.g.* for approximately 1% accuracy we need 100 terms for $p = 0$, but only 10 terms for $p = 1$) and for checking calculations (*e.g.* an erroneous contribution to a Fourier coefficient can be rapidly identified if it does not have the typical rate of decay for large n — such mistakes often occur when multiple integration by parts are required to evaluate a Fourier coefficient).

Gibb's phenomenon

- This is the persistent overshoot in Example 2 near a jump discontinuity. It happens whenever a jump discontinuity exists.
- As the number of terms in the partial sum tends to ∞, the width of the overshoot region tends to 0 (by the Fourier Convergence Theorem), while the total height of the overshoot region approaches $\gamma|f(x_+) - f(x_-)|$, where

$$\gamma = \frac{1}{\pi}\int_{\pi}^{\pi} \frac{\sin x}{x}\,\mathrm{d}x \approx 1.18,$$

i.e. approximately a 9% overshoot top and bottom. This is awful for approximation purposes!

Functions of any period

- Suppose now $f : \mathbb{R} \to \mathbb{R}$ is a periodic function of period $2L$, where L is a positive number, not necessarily equal to π.
- We want to develop the analogous results for the Fourier series for $f(x)$. Since this will involve a series in the trigonometric functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$, where n is a positive integer, we make the transformation

$$x = \frac{LX}{\pi}, \qquad f(x) = g(X)$$

which defines a new function $g : \mathbb{R} \to \mathbb{R}$.

- For $X \in \mathbb{R}$, it follows that

$$g(X + 2\pi) = f\left(\frac{L}{\pi}(X + 2\pi)\right) = f\left(\frac{LX}{\pi} + 2L\right) = f\left(\frac{LX}{\pi}\right) = g(X),$$

where we used the fact that $g(X) = f(LX/\pi)$ in the first equality; the fact that f is $2L$-periodic in the third equality; and the fact that $f(x) = g(LX/\pi)$ in the third equality. Thus, g is 2π-periodic, and we can use the transformation to derive the Fourier theory for f from that for g above.

- In particular, if we can write

$$g(X) = \frac{a_0}{2} + \sum_{n=1}^{1}\big(a_n\cos(nX) + b_n\sin(nX)\big),$$

so that the Fourier coefficients a_n and b_n exist, then

$$a_n = \frac{1}{\pi}\int_{\pi}^{\pi} g(X)\cos(nX)\,\mathrm{d}X = \frac{1}{\pi}\int_{L}^{L} g\left(\frac{\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x = \frac{1}{L}\int_{L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

and

$$b_n = \frac{1}{\pi}\int_{\pi}^{\pi} g(X)\sin(nX)\,\mathrm{d}X = \frac{1}{\pi}\int_{L}^{L} g\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)\frac{\pi}{L}\,\mathrm{d}x = \frac{1}{L}\int_{L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x\,.$$

- **Definition:** Suppose f is $2L$-periodic and such that the Fourier coefficients a_n and b_n exist. Then we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

where $\sim$ means the RHS is the Fourier series for f, regardless of whether or not it converges to f.

- **Fourier Convergence Theorem (FCT):** Let $f : \mathbb{R} \to \mathbb{R}$ be $2L$-periodic, with f and f' piecewise continuous on $(-L, L)$. Then, the Fourier coefficients a_n and b_n exist, and

$$\frac{1}{2}\big(f(x_+) + f(x_-)\big) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \quad \text{for} \quad x \in \mathbb{R}.$$

Cosine and sine series

- **Definition:** The even $2L$-periodic extension $f_e : \mathbb{R} \to \mathbb{R}$ of $f : [0, L] \to \mathbb{R}$ is defined by

$$f_e(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L\,, \\ f(-x) & \text{for } -L < x < 0\,, \end{cases} \qquad \text{with } f_e(x + 2L) = f_e(x) \text{ for } x \in \mathbb{R}.$$

The Fourier cosine series for $f : [0, L] \to \mathbb{R}$ is the Fourier series for f_e, *i.e.*

$$f_e(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right),$$

where

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad (n \in \mathbb{N}).$$

- **Definition:** The odd $2L$-periodic extension $f_o : \mathbb{R} \to \mathbb{R}$ of $f : [0, L] \to \mathbb{R}$ is defined by

$$f_o(x) = \begin{cases} f(x) & \text{for } 0 \le x \le L\,, \\ -f(-x) & \text{for } -L < x < 0\,, \end{cases} \qquad \text{with } f_o(x + 2L) = f_o(x) \text{ for } x \in \mathbb{R}.$$

The Fourier sine series for $f : [0, L] \to \mathbb{R}$ is the Fourier series for f_o, *i.e.*

$$f_o(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right),$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \quad (n \in \mathbb{N}\backslash\{0\}).$$

- **Remarks:**

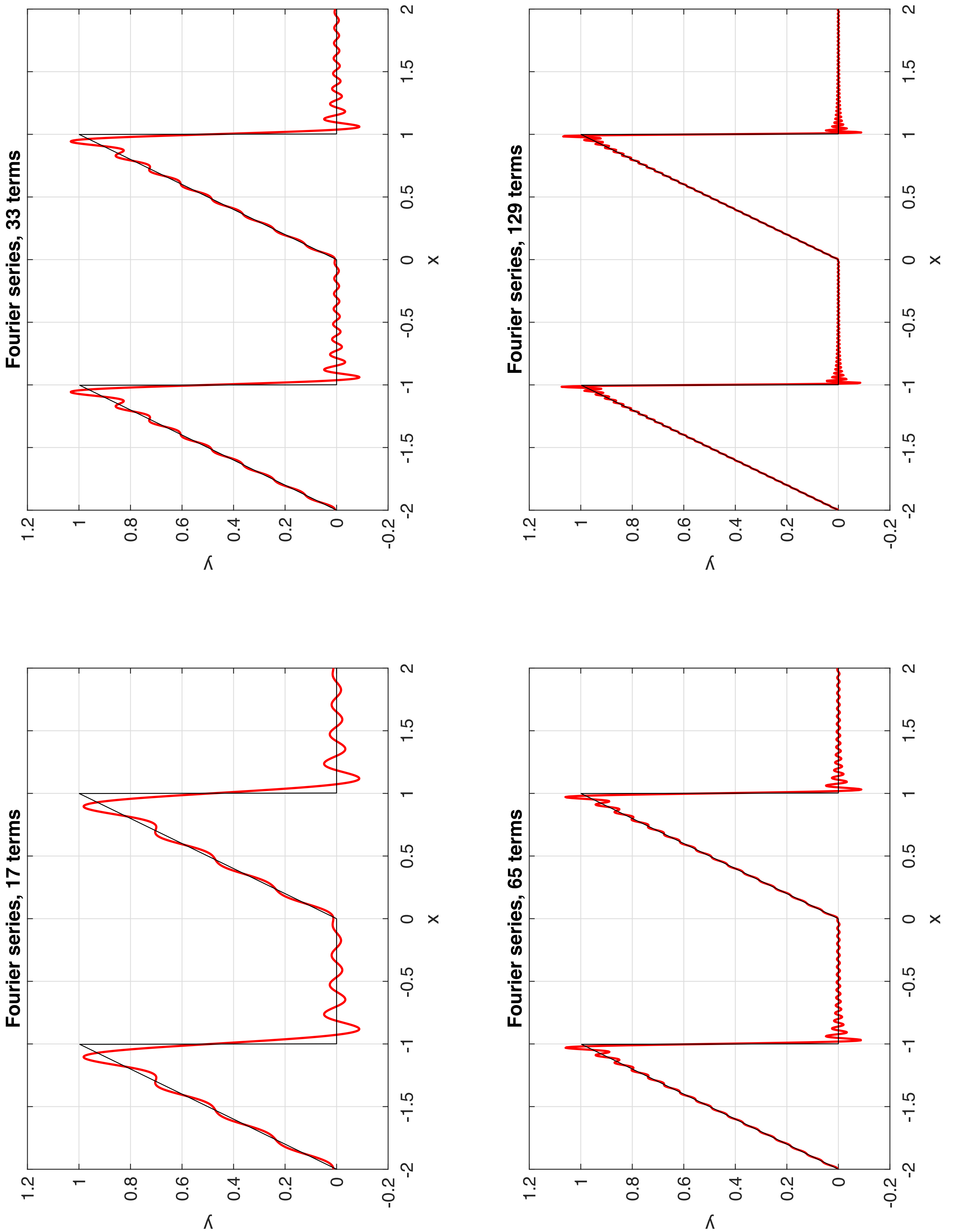
  (1) Note that $f_o(x)$ is odd for $x/L \in \mathbb{R}\backslash\mathbb{Z}$ and odd (on $\mathbb{R}$) if and only if $f(0) = f(L) = 0$.

  (2) Note that if f is continuous on $[0, L]$ and f' piecewise continuous on $(0, L)$, then the Fourier Convergence Theorem implies that

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f_e(x) \text{ for } x \in \mathbb{R}\,,$$

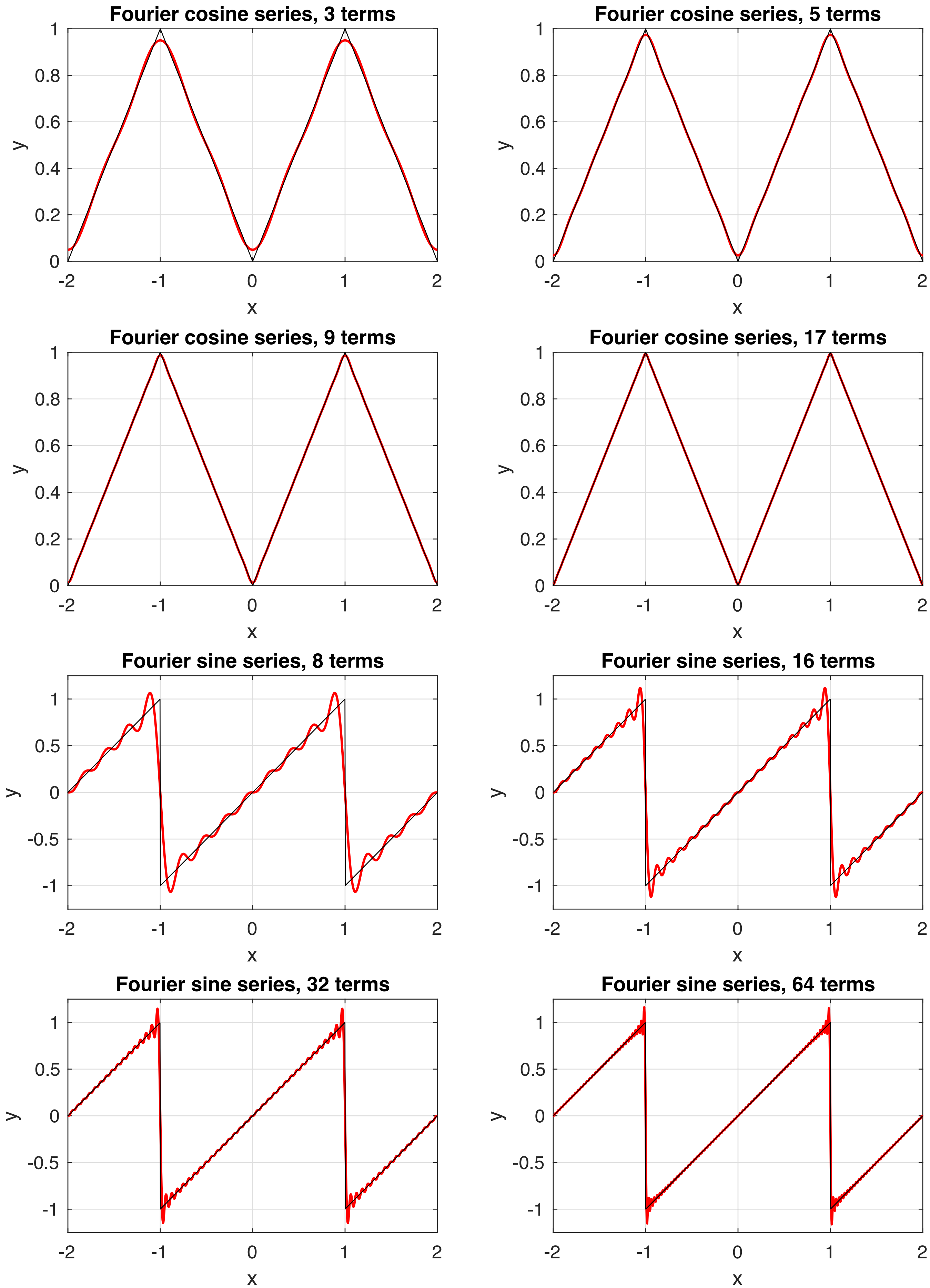
$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \begin{cases} f_o(x) & \text{for } x/\pi \in \mathbb{R}\backslash\mathbb{Z}, \\ 0 & \text{for } x/\pi \in \mathbb{R}\backslash\mathbb{Z}. \end{cases}$$

Example 3



Fourier series, 17 terms
Fourier series, 33 terms
Fourier series, 65 terms
Fourier series, 129 terms
x
y
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
-0.2
0.2
0.4
0.6
0.8
1.2

Example 4


Fourier cosine series, 3 terms
Fourier cosine series, 5 terms
Fourier cosine series, 9 terms
Fourier cosine series, 17 terms
Fourier sine series, 8 terms
Fourier sine series, 16 terms
Fourier sine series, 32 terms
Fourier sine series, 64 terms
x
y