Heat conduction in a finite rod

Consider the initial boundary value problem (IBVP) for the temperature $T(x,t)$ in a rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L, \ t > 0, \tag{1}$$

with the boundary conditions

$$T(0,t) = 0, \quad T(L,t) = 0 \quad \text{for} \quad t > 0, \tag{2}$$

and the initial condition

$$T(x,0) = f(x) \quad \text{for} \quad 0 < x < L, \tag{3}$$

where the initial temperature profile $f(x)$ is given.

Fourier's method

We will construct a solution using Fourier's method, which consists of the following three steps:

(I) Use the method of separation of variables to find the countably infinite set of nontrivial separable solutions satisfying the partial differential equation (1) and boundary conditions (2), each containing an arbitrary constant.

(II) Use the principle of superposition — that the sum of any number of solutions of a linear problem is also a solution (assuming convergence) — to form the general series solution that is the infinite sum of the separable solutions of the partial differential equation and boundary conditions.

(III) Use the theory of Fourier series to determine the constants in the general series solution for which it satisfies the initial condition (3).

Remarks

- (1) and (2) are linear since, if T_1 and T_2 satisfy them, then so does $\alpha T_1 + \beta T_2$ for all $\alpha, \ \beta \in \mathbb{R}$.
- To verify that the resulting series is actually a solution of the PDE, we need it to converge sufficiently rapidly that T_t and T_{xx} can be computed by termwise differentiation — we largely gloss over such issues.

Remarks on the series solution constructed in the lecture

- The integral expressions for the Fourier coefficients may be derived via the orthogonality relations

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{L}{2}\delta_{mn} \qquad (m, \, n \in \mathbb{N} \setminus \{0\})$$

by assuming that the orders of summation and integration may be interchanged, as follows:

$$\begin{aligned}
\frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x &= \frac{2}{L}\int_0^L \sum_{m=1}^{\infty} b_m \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\
&= \sum_{m=1}^{\infty} b_m \frac{2}{L}\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\
&= \sum_{m=1}^{\infty} b_m \delta_{mn} \\
&= b_n \qquad \text{for} n \in \mathbb{N} \setminus \{0\}.
\end{aligned}$$

- If f and f' are piecewise continuous on $(0, L)$, then the Fourier Convergence Theorem implies that the Fourier sine series for f converges to $(f(x_+) + f(x_-))/2$ for $x \in (0, L)$ and to 0 for $x = 0, L$. Thus, Fourier's method can even handle jump discontinuities in the initial temperature profile, with the caveat that the truncated series solution would exhibit Gibb's phenomenon at time $t = 0$.

- It can be rigorously proven (using methods from Prelims Analysis) that for such initial conditions, the series solution converges and is a solution of the initial boundary value problem (1)–(3). Since $T_n(x, t)$ decays exponentially as $n \to \infty$ for fixed $0 < x < L$ and fixed $t > 0$, it may also be shown that all partial derivatives with respect to x and t exist and may be derived by termwise partial differentiation of the series!

- There are several important implications of the last two remarks:
  - the heat equation *smoothes* out instantaneously even irregular initial temperature profiles;
  - as soon as $t > 0$, most of the high frequency terms $T_n(x, t)$ for $n \gg 1$ will be extremely small, so that the solution may be well approximated by only a handful of terms;
  - the temperature tends to zero exponentially quickly $\kappa t/L^2 \to \infty$, *i.e.* on the timescale of heat conduction, with the thermal energy initially stored in the rod being conducted out of the ends of the rod on this timescale.

- Consider the initial profile given by

$$f(x) = \begin{cases} T^* & \text{for } L_1 < x < L_2\,, \\ 0 & \text{otherwise,} \end{cases}$$

where T^*, L_1 and L_2 are constants, for which the Fourier coefficients are given by

$$b_n = \frac{2}{L}\int_{L_1}^{L_2} T^* \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{2T^*}{n\pi}\left(\cos\left(\frac{n\pi L_1}{L}\right) - \cos\left(\frac{n\pi L_2}{L}\right)\right)$$

for positive integers n. We plot below snapshots of the partial sums of the truncated series solution with 100 terms for $L_1/L = 0.2$, $L_2/L = 0.4$ and $100\kappa t/L^2 = 0$, 0.25, 0.5, 1, 2, 4, 8, 16 and 32, which illustrates all of the main features of the solution discussed above.