Fourier Series and PDEs Problem Sheet 1

1. (a) Let f be a periodic function of period 2π. By making the substitution $s = 2\pi + t$ in one of the integrals in the identity

$$\int_{\alpha}^{\alpha+2\pi} f(s)\,\mathrm{d}s = \int_{\alpha}^{\pi} f(s)\,\mathrm{d}s + \int_{\pi}^{\alpha+2\pi} f(s)\,\mathrm{d}s,$$

show that, for any real α,

$$\int_{\alpha}^{\alpha+2\pi} f(s)\,\mathrm{d}s = \int_{-\pi}^{\pi} f(s)\,\mathrm{d}s.$$

(b) Let g be an even function with derivative g' and suppose that the function G is defined by

$$G(x) = \int_0^x g(s)\,\mathrm{d}s.$$

(i) Use the chain rule to show that g' is an odd function.
(ii) By making the substitution $s = -t$, show that G is an odd function.
(iii) Deduce that, for any real α,

$$\int_{-\alpha}^{\alpha} g(s)\,\mathrm{d}s = 2\int_0^{\alpha} g(s)\,\mathrm{d}s.$$

(c) Show that if h is an odd function then, for any real α,

$$\int_{-\alpha}^{\alpha} h(s)\,\mathrm{d}s = 0.$$

2. (a) Establish the *orthogonality relations* given by

$$\begin{aligned}
\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x &= \pi\delta_{mn},\\
\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x &= 0,\\
\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x &= \pi\delta_{mn},
\end{aligned}$$

where m and n are positive integers and δ_{mn} is Kronecker's delta.
[*You may find it helpful to use the identities* $\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$.]

(b) Let f be a continuous periodic function of period 2π. Suppose that f may be expressed in the form of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big).$$

Assuming that the orders of summation and integration may be interchanged, derive
(i) integral expressions over $[-\pi, \pi]$ for the Fourier coefficients a_n and b_n;
(ii) *Parseval's identity* in the form

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,\mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right).$$

[*For (ii) you may find it helpful to integrate the product of* f *and the Fourier series for* f.]

3. The periodic function f of period 2π is defined by

$$f(x) = x\sin(px) \quad \text{for } -\pi < x \le \pi,$$

where p is a positive integer.

(a) Is f even or odd? Use integration by parts to show that if m is a non-zero integer then

$$\int_0^{\pi} x\sin(mx)\,\mathrm{d}x = \frac{\pi(-1)^{m+1}}{m}.$$

Hence, or otherwise, show that the Fourier series for f is given by

$$f(x) \sim \frac{(-1)^{p+1}}{p} - \frac{\cos(px)}{2p} + \sum_{\substack{n=1\\ n\neq p}}^{\infty} \frac{2p(-1)^{n+p}}{n^2-p^2}\cos(nx).$$

[*You may find it helpful to use the identity* $2\sin A\cos B = \sin(A+B) + \sin(A-B)$.]

(b) Given that the Fourier series converges to $f(x)$ for $x = 0$ and $x = \pi$, deduce that

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} = \frac{1}{4} \qquad \text{and} \qquad \sum_{n=3}^{\infty} \frac{1}{n^2-4} = \frac{25}{48}.$$

(c) Given that Parseval's identity holds for f, evaluate the sum

$$\sum_{\substack{n=1\\ n\neq p}}^{\infty} \frac{1}{(n^2-p^2)^2}.$$