Fourier Series and PDEs Problem Sheet 2

1. Let f be a continuous periodic function of period $2L$. For positive integers N, denote by

$$S_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

the partial sums of the Fourier series for f, and let T_N be the sum defined by

$$T_N(x) = \frac{A_0}{2} + \sum_{n=1}^{N} \left(A_n \cos\left(\frac{n\pi x}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \right),$$

where A_n and B_n are arbitrary real constants.

(a) State integral expressions over $[-L, L]$ for the Fourier coefficients a_n and b_n.

(b) Show that

$$\frac{1}{L}\int_{-L}^{L} f(x) T_N(x)\,\mathrm{d}x = \frac{a_0 A_0}{2} + \sum_{n=1}^{N} \left(a_n A_n + b_n B_n \right).$$

Hence write down similar expressions for

$$\frac{1}{L}\int_{-L}^{L} T_N(x)^2\,\mathrm{d}x, \qquad \frac{1}{L}\int_{-L}^{L} f(x) S_N(x)\,\mathrm{d}x \qquad \text{and} \qquad \frac{1}{L}\int_{-L}^{L} S_N(x)^2\,\mathrm{d}x.$$

[*You may find it helpful to note that (i) if $f = T_N$ then $a_n = A_n$ and $b_n = B_n$ for $n \le N$ and (ii) if $T_N = S_N$ then $A_n = a_n$ and $B_n = b_n$ for $n \le N$.*]

(c) Deduce that

$$\mathcal{E}(T_N) - \mathcal{E}(S_N) = \frac{(A_0 - a_0)^2}{2} + \sum_{n=1}^{N} \left((A_n - a_n)^2 + (B_n - b_n)^2 \right),$$

where the *mean-squared error* in the approximation of f by T_N is defined by

$$\mathcal{E}(T_N) = \frac{1}{L}\int_{-L}^{L} \left(T_N(x) - f(x) \right)^2 \mathrm{d}x.$$

Hence write down the values of the constants A_n and B_n for which $\mathcal{E}(T_N)$ is minimized for each positive integer N.

2. The periodic function f of period $2L$ is defined by

$$f(x) = \begin{cases} 1 + x/L & \text{for } -L < x \le 0, \\ 0 & \text{for } 0 < x \le L. \end{cases}$$

(a) Sketch the graph of $f(x)$ for $x \in (-2L, 2L]$, indicating any point x at which the Fourier series for f does not converge to $f(x)$ and showing the values to which the Fourier series does converge at these points.

(b) Show that the Fourier series for f is given by

$$f(x) \sim \frac{1}{4} + \sum_{m=1}^{\infty} \left(\frac{2}{(2m-1)^2\pi^2} \cos\left(\frac{(2m-1)\pi x}{L}\right) - \frac{1}{m\pi} \sin\left(\frac{m\pi x}{L}\right) \right).$$

(c) By considering the value of the series at $x = L/2$ and at $x = 0$, deduce that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4} \qquad \text{and} \qquad \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

3. (a) The function f is defined by

$$f(x) = \exp(x/L) \quad \text{for } 0 \leq x \leq L.$$

(i) Evaluate the Fourier cosine and sine series for f on $[0, L]$.

(ii) For each series sketch the graph of the function to which it converges on $(-2L, 2L]$.

(iii) Describe three ways in which the truncated cosine series for f is a better approximation of f on $[0, L]$ than the truncated sine series for f.

(b) Give, with justification, an example of a polynomial that is better approximated on $[0, L]$ by its truncated sine series than its truncated cosine series.

[*You may find helpful the* MATLAB *file* `FS_PDE_Sheet2_Q3.m` *on the course website.*]