Fourier Series and PDEs Problem Sheet 3

1. Consider an isotropic conducting rod of constant cross-sectional area A lying along the x-axis. The rod has density ρ, specific heat c, thermal conductivity k and thermal diffusivity $\kappa = k/(\rho c)$, all of these material parameters being constant. Let $T(x,t)$ denote the temperature and $q(x,t)$ the heat flux in the positive x-direction, where t is time. The lateral surfaces of the rod are insulated and heat is supplied at a prescribed rate $Q(x,t)$ per unit volume per unit time. You may assume that conservation of energy in a fixed section $a \le x \le a+h$ of the rod is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(A\int_a^{a+h} \rho c T(x,t)\,\mathrm{d}x\right) = Aq(a,t) - Aq(a+h,t) + A\int_a^{a+h} Q(x,t)\,\mathrm{d}x.$$

   (a) What is the physical significance of each of the four terms in this equality?

   (b) Write down *Fourier's law* and deduce that $T(x,t)$ satisfies the inhomogeneous heat equation

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(x,t),$$

   stating any assumptions that you make concerning the smoothness of T and Q.

   (c) Write down the dimensions of each of A, ρ, c, x, t, T and q in terms of the SI units m, s, Kg, K and J. Deduce that the dimensions of Q, k and κ are given by

$$[Q] = \mathrm{J\,m^{-3}\,s^{-1}}, \quad [k] = \mathrm{J\,K^{-1}\,m^{-1}\,s^{-1}}, \quad [\kappa] = \mathrm{m^2\,s^{-1}}.$$

2. Consider the temperature $T(x,t)$ defined by

$$T(x,t) = \frac{T^* L}{\sqrt{4\pi\kappa t}}\exp\left(-\frac{x^2}{4\kappa t}\right) \quad \text{for } -\infty < x < \infty,\ t > 0,$$

where the thermal diffusivity κ, temperature T^* and length scale L are positive constants.

   (a) Verify that the expression for $T(x,t)$ is dimensionally correct and show that it satisfies the heat equation

$$\frac{\partial T}{\partial t} = \kappa\frac{\partial^2 T}{\partial x^2} \quad \text{for } -\infty < x < \infty,\ t > 0.$$

   (b) By making the change of variable $x = \sqrt{4\kappa t}\,\eta$, show that

$$\int_{-\infty}^{\infty} T(x,t)\,\mathrm{d}x = T^* L \quad \text{for } t > 0.$$

   [*You may quote the fact that* $\int_{-\infty}^{\infty} e^{-\eta^2}\,d\eta = \sqrt{\pi}$.]

   (c) Sketch the graph of T versus x, at fixed t, for (i) $t \ll L^2/\kappa$ and (ii) $t \gg L^2/\kappa$. Use the same axes for each graph.

3. Consider the initial boundary value problem for the temperature $T(x,t)$ in a rod of length L given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L, \ t > 0,$$

with the boundary conditions $T(0,t) = 0$ and $T(L,t) = 0$ for $t > 0$ and the initial condition $T(x,0) = T^* \, x \, (L - x) \, / L^2$ for $0 < x < L$, where the thermal diffusivity κ and temperature T^* are positive constants.

(a) Use the method of separation of variables to show that, if $T(x,t) = F(x)G(t)$ is a nontrivial solution of the heat equation satisfying the boundary conditions, then for some constant λ

$$-F'' = \lambda F \quad \text{for} \quad 0 < x < L \quad \text{with} \quad F(0) = 0, \ F(L) = 0.$$

Determine all real values of λ for which there is a nontrivial solution of the boundary value problem for F and the corresponding separable solutions for T.

(b) Use the principle of superposition and the theory of Fourier series to derive the series solution for $T(x,t)$ given by

$$T(x,t) = \sum_{m=0}^{\infty} \frac{8T^*}{(2m+1)^3 \pi^3} \sin\left(\frac{(2m+1)\pi x}{L}\right) \exp\left(-\frac{(2m+1)^2 \pi^2 \kappa t}{L^2}\right).$$

[*You may assume that the orders of summation and integration may be interchanged as necessary and the orthogonality relations*

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \, \mathrm{d}x = \frac{L}{2} \delta_{mn},$$

where m and n are positive integers and δ_{mn} is Kronecker's delta.]

(c) Verify that the series solution is dimensionally correct. Explain why the smoothness of the odd $2L$-periodic extension for $T(x,0)$ is consistent with the rate of convergence of its Fourier sine series.