Fourier Series and PDEs Problem Sheet 4

1. Consider the initial boundary value problem for the temperature $T(x,t)$ in a rod of length L and thermal diffusivity κ given by the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the boundary conditions $T_x(0,t) = 0$ and $T_x(L,t) = 0$ for $t > 0$ and the initial condition $T(x,0) = T^*\, x\,(L-x)\,/L^2$ for $0 < x < L$, where T^* is a positive constant.

   (a) Show that the solution $T(x,t)$ is uniquely determined.

   (b) Use the method of separation of variables, the principle of superposition and the theory of Fourier series to derive the series solution given by

$$T(x,t) = \frac{T^*}{6} - \sum_{m=1}^{\infty} \frac{T^*}{m^2\pi^2} \cos\left(\frac{2m\pi x}{L}\right) \exp\left(-\frac{4m^2\pi^2\kappa t}{L^2}\right).$$

   (c) What is the behaviour of the temperature $T(x,t)$ in the limit as $t \to \infty$?

[*In part (b) you may assume that the orders of summation and integration may be interchanged as necessary and the identities*

$$\int_0^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x = \frac{L}{2}\delta_{mn}, \quad \int_0^L x(L-x)\cos\left(\frac{n\pi x}{L}\right) \mathrm{d}x\, \mathrm{d}x = -\frac{L^3(1+(-1)^n)}{n^2\pi^2},$$

where m and n are positive integers and δ_{mn} is Kronecker's delta.]

2. (a) Let κ and ω be positive constants. Show that the heat equation

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2}$$

   has complex-valued solutions of the form $F(x)\mathrm{e}^{\mathrm{i}\omega t}$ provided

$$\kappa F'' = \mathrm{i}\omega F.$$

   Hence find F if $F'(x) \to 0$ as $x \to \infty$ and $F(0) = T_1$, where T_1 is a positive constant. [*You may assume that the roots of $\lambda^2 = \mathrm{i}\omega/\kappa$ are $\lambda = \pm(1+\mathrm{i})\sqrt{\omega/2\kappa}$.*]

   (b) Now let $T(x,t) = T_0 + \mathrm{Re}\left(F(x)\mathrm{e}^{\mathrm{i}\omega t}\right)$, where T_0 is a real constant. Verify that

$$T(x,t) = T_0 + T_1 \exp\left(-\sqrt{\frac{\omega}{2\kappa}}x\right) \cos\left(\omega t - \sqrt{\frac{\omega}{2\kappa}}x\right),$$

   and explain why $T(x,t)$ is a solution of the heat equation for which $T_x(x,t) \to 0$ as $x \to \infty$ and $T(0,t) = T_0 + T_1 \cos(\omega t)$.

   (c) A *root cellar* is used to store crops, ideally by keeping them as cool as possible in the summer, but as warm as possible in the winter. Consider a root cellar buried in soil of thermal diffusivity $\kappa = 10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. Use the temperature profile in part (b) to predict

      (i) the shallowest ideal depth of the root cellar;

      (ii) the factor by which the amplitude of the temperature oscillations at ground level are reduced at the shallowest ideal depth.

3. Consider the initial boundary value problem for the temperature $T(x,t)$ in a rod of length L given by the inhomogeneous heat equation

$$\rho c \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + Q(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the boundary conditions $T(0,t) = \phi(t)$ and $T(L,t) = \psi(t)$ for $t > 0$ and the initial condition $T(x,0) = f(x)$ for $0 < x < L$, where ρ, c and k are positive constants and the functions $Q(x,t)$, $\phi(t)$, $\psi(\mathrm{t})$ and $f(x)$ are given.

(a) Let

$$T(x,t) = \phi(t)\left(1 - \frac{x}{L}\right) + \psi(t)\frac{x}{L} + U(x,t).$$

Determine the functions $\widetilde{Q}$ and $\widetilde{f}$ for which U satisfies the initial boundary value problem given by

$$\rho c \frac{\partial U}{\partial t} = k \frac{\partial^2 U}{\partial x^2} + \widetilde{Q}(x,t) \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with $U(0,t) = U(L,t) = 0$ for $t > 0$ and $U(x,0) = \widetilde{f}(x)$ for $0 < x < L$.

(b) By considering your answer to question 3 of sheet 3, write down the solution for $U(x,t)$ in the special case in which $\widetilde{Q}(x,t) = 0$ for $0 < x < L$, $t > 0$.

(c) Consider now the case in which $\widetilde{Q}$ is not identically zero. Suppose that $U(x,t)$ and $\widetilde{Q}(x,t)$ may be expanded as the Fourier sine series

$$U(x,t) = \sum_{n=1}^{\infty} U_n(t) \sin\left(\frac{n\pi x}{L}\right), \qquad \widetilde{Q}(x,t) = \sum_{n=1}^{\infty} \widetilde{Q}_n(t) \sin\left(\frac{n\pi x}{L}\right),$$

where the Fourier coefficients are given by

$$U_n(t) = \frac{2}{L}\int_0^L U(x,t) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x, \qquad \widetilde{Q}_n(t) = \frac{2}{L}\int_0^L \widetilde{Q}(x,t) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x.$$

(i) By differentiating $U_n(t)$ under the integral sign, using the heat equation and integrating by parts, show that

$$\rho c \frac{\mathrm{d}U_n}{\mathrm{d}t} + \frac{kn^2\pi^2}{L^2} U_n = \widetilde{Q}_n \quad \text{for} \quad t > 0.$$

Use the initial condition for U to write down the initial condition for U_n.

(ii) Explain without any further calculations how to determine the temperature $T(x,t)$ given the functions $Q(x,t)$, $\phi(t)$, $\psi(t)$ and $f(x)$.

(iii) What are the advantages of expanding U as a Fourier sine series rather than T?