Fourier Series and PDEs Problem Sheet 5

1. Consider the small transverse vibrations of a homogeneous extensible elastic string stretched initially along the x-axis to a constant line density ρ. A point initially at $x\boldsymbol{i}$ is displaced transversely at time t to the point $\boldsymbol{r}(x,t) = x\boldsymbol{i} + y(x,t)\boldsymbol{j}$, where the slope y_x is everywhere small. The string offers no resistance to bending in the sense that the string to the right of the point $\boldsymbol{r}(x,t)$ exerts at that point a tangential force $T(x,t)\boldsymbol{\tau}(x,t)$ on the string to the left, where $T(x,t)$ is the tension and $\boldsymbol{\tau}(x,t) = \boldsymbol{r}_x/|\boldsymbol{r}_x|$ is the unit tangent vector pointing in the positive x-direction. The string is subject to a gravitational field $-g\boldsymbol{j}$ and air resistance exerts a force $-\gamma\boldsymbol{r}_t$ per unit length, where the acceleration due to gravity g and the drag coefficient γ are positive constants. You may assume that Newton's second law for the piece of string in the fixed region $a \le x \le a+h$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_a^{a+h} \rho\boldsymbol{r}_t\,\mathrm{d}x\right) = T(a+h,t)\boldsymbol{\tau}(a+h,t) - T(a,t)\boldsymbol{\tau}(a,t) - \int_a^{a+h} \rho g\boldsymbol{j}\,\mathrm{d}x - \int_a^{a+h} \gamma\boldsymbol{r_t}\,\mathrm{d}x.$$

   (a) State the physical significance of each of the terms in this equality, illustrating your answer with a sketch showing the forces acting on the piece of string in $a \le x \le a+h$.

   (b) Show that, to a first approximation for $|y_x| \ll 1$,

      (i) the tension T is spatially uniform in the sense that
$$\frac{\partial T}{\partial x} = 0;$$

      (ii) the transverse displacement $y(x,t)$ satisfies the forced and damped wave equation
$$\rho\frac{\partial^2 y}{\partial t^2} = T(t)\frac{\partial^2 y}{\partial x^2} - \rho g - \gamma\frac{\partial y}{\partial t}.$$

   You should state any assumptions that you make concerning the smoothness of T and y.

   (c) Suppose that T is constant and let the wave speed $c = \sqrt{T/\rho}$. If the wave equation is nondimensionalized by scaling $x = L\,\widehat{x}$, $t = L\,\widehat{t}/c$ and $y(x,t) = H\,\widehat{y}\,(\widehat{x},\widehat{t}\,)$, where L and H are typical horizontal and transverse length scales, show that

$$\frac{\partial^2\widehat{y}}{\partial\widehat{t}^2} = \frac{\partial^2\widehat{y}}{\partial\widehat{x}^2} - \alpha - \beta\frac{\partial\widehat{y}}{\partial\widehat{t}},$$

   where α and β are dimensionless parameters that you should determine. Under what conditions are the effects of gravity and air resistance negligible?

2. Consider the transverse displacement $y(x,t)$ defined by

$$y(x,t) = \frac{1}{2}\big(h(x+ct) + h(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} u(s)\,\mathrm{d}s, \qquad (\star)$$

where the wave speed c is a positive constant, the given function h is twice differentiable and the given function u is differentiable.

   (a) Show that $y(x,t)$ satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad -\infty < x < \infty,\ t > 0,$$

   and the initial conditions $y(x,0) = h(x)$ and $y_t(x,0) = u(x)$ for $-\infty < x < \infty$.

   (b) Consider the case in which

$$h(x) = \begin{cases} H\left(1 - \dfrac{x^2}{L^2}\right)^4 & \text{for } |x| \le L, \\ 0 & \text{for } |x| > L, \end{cases} \qquad u(x) = 0 \quad \text{for} \quad -\infty < x < \infty.$$

   where H and L are positive constants.

      (i) Sketch the graphs of $h(x-ct)$ and $h(x+ct)$ versus x for $t = 0$, $t = L/c$ and $t = 2L/c$.

      (ii) Hence sketch the graph of $y(x,t)$ verses x for $t = 0$, $t = L/c$ and $t = 2L/c$.

3. Consider the initial boundary value problem for the small transverse displacement $y(x,t)$ of an elastic string given by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for} \quad 0 < x < L,\ t > 0,$$

with the boundary conditions $y(0,t) = 0$ and $y(L,t) = 0$ for $t > 0$ and the initial conditions $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$ for $0 < x < L$, where the wave speed c and length L are positive constants and the given functions f and g are continuous on $[0,L]$ and have piecewise continuous derivatives on $(0,L)$, with $f(0) = f(L) = g(0) = g(L) = 0$.

(a) Use the method of separation of variables to show that the *normal modes* are given for positive integers n by

$$y_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right),$$

where a_n and b_n are constants. Write down integral expressions over $[0,L]$ for the constants a_n and b_n for which the initial conditions are satisfied by the series solution

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t).$$

[*You may quote the formulae for the Fourier coefficients of a sine series.*]

(b) Let a, ℓ and v be real constants with $0 < \ell < L$. Find the series solution for $y(x,t)$ for

(i) a plucked string for which

$$f(x) = \begin{cases} ax/\ell & \text{for } 0 < x < \ell, \\ a(L-x)/(L-\ell) & \text{for } \ell < x < L, \end{cases} \qquad g(x) = 0;$$

(ii) a flicked string for which

$$f(x) = 0, \qquad g(x) = \begin{cases} v & \text{for } |x - L/2| \le \ell/2, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Show that the series solution in part (a) may be written in the form ($\star$) of question 2 for suitable functions h and u that you should determine.

[*You may quote the identities*

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right) = \frac{1}{2}\left(\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right)\right),$$

$$\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right) = \frac{n\pi}{2L}\int_{x-ct}^{x+ct} \sin\left(\frac{n\pi s}{L}\right) \mathrm{d}s,$$

and you may assume that the orders of summation and integration may be interchanged as necessary.]