Fourier Series and PDEs Problem Sheet 7

1. (a) By introducing new independent variables $\xi = x - ct$ and $\eta = x + ct$, show that if $y(x,t)$ is a solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

in which the wave speed c is a positive constant, then there are functions F and G such that

$$y(x,t) = F(x-ct) + G(x+ct).$$

Deduce that, if y satisfies the given initial conditions $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$ for $-\infty < x < \infty$, then the solution is given by *D'Alembert's formula*

$$y(x,t) = \frac{1}{2}\big(f(x-ct) + f(x+ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,\mathrm{d}s.$$

(b) Consider the case in which

$$f(x) = 0 \quad \text{and} \quad g(x) = \frac{v\ell}{\ell + |x|},$$

where v and ℓ are positive constants. Show that if $0 < ct < x$ then the solution is given by

$$y(x,t) = \frac{v\ell}{2c}\ln\left(\frac{\ell + x + ct}{\ell + x - ct}\right).$$

Find the solution if $0 < x < ct$. Hence, write down the solution if $x < 0$ and $t > 0$.

(c) Consider the case in which

$$f(x) = \begin{cases} \varepsilon(a - |x|) & \text{for } |x| \le a, \\ 0 & \text{for } |x| > a, \end{cases} \qquad g(x) = 0,$$

where ε and a are positive constants. Find the solution $y(x,t)$ at every time $t > 0$ and sketch the graph of y versus x at the times $t = 0$, $t = a/2c$, $t = a/c$ and $t = 3a/2c$.

(d) Consider the case in which

$$f(x) = 0, \qquad g(x) = \begin{cases} v & \text{for } |x| \le a, \\ 0 & \text{for } |x| > a, \end{cases}$$

where v and a are positive constants. Find the solution $y(x,t)$ at every time $t > 0$ and sketch the graph of y versus x at the times $t = 0$, $t = a/2c$, $t = a/c$ and $t = 3a/2c$.

(e) Consider the case in which $f(x) = 0$ for $-\infty < x < \infty$, $g(x) = vx/a$ for $a < x < 2a$, $g(x) = -vx/a$ for $-2a < x < -a$, but $g(x)$ is *not prescribed* for other values of x, where v and a are positive constants. Label in a sketch the parts of the (x,t)-plane where $y(x,t)$ can be determined for $t > 0$, and obtain an expression for $y(x,t)$ in each such region.

2. An infinite straight metal rod has a square cross-section whose sides are of length L. The temperature $T(x,y)$ in each cross-section satisfies the boundary value problem given by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < L,\ 0 < y < L,$$

with the boundary conditions

$$\begin{aligned} T(0,y) &= 0 && \text{for} \quad 0 < y < L, && (1) \\ T(L,y) &= 0 && \text{for} \quad 0 < y < L, && (2) \\ T(x,0) &= 0 && \text{for} \quad 0 < x < L, && (3) \\ T(x,L) &= T^* && \text{for} \quad 0 < x < L, && (4) \end{aligned}$$

where T^* is a positive constant.

(a) Use the method of separation of variables and the principle of superposition to derive the general series solution satisfying Laplace's equation and the boundary conditions (1)–(2) given by

$$T(x,y) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(A_n \cosh\left(\frac{n\pi y}{L}\right) + B_n \sinh\left(\frac{n\pi y}{L}\right)\right),$$

where A_n and B_n are constants. Determine the constants A_n and B_n for which the general series solution satisfies the boundary conditions (3)–(4).

[*You may quote the formulae for the Fourier coefficients of a sine series.*]

(b) By considering three similar problems, show that $T = T^*/4$ at the centre of the square.