Fourier Series and PDEs Problem Sheet 8

1. An infinite metal slab of constant thermal conductivity k has cross-section that is a semi-infinite strip of width L. The temperature $T(x, y)$ in each cross-section satisfies the boundary value problem given by Laplace's equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad \text{for} \quad 0 < x < L,\ y > 0,$$

with the boundary conditions

$$\begin{aligned} T_x(0, y) &= 0 && \text{for} \quad y > 0, \\ T_x(L, y) &= 0 && \text{for} \quad y > 0, \\ \lim_{y \to \infty} T_y(x, y) &= \frac{q^*}{k} && \text{for} \quad 0 < x < L, \\ T(x, 0) &= T^* \sin\left(\frac{\pi x}{L}\right) && \text{for} \quad 0 < x < L, \end{aligned}$$

where q^* and T^* are positive constants.

   (a) Use the method of separation of variables, the principle of superposition and the theory of Fourier series to derive the series solution given by

$$T(x, y) = \frac{2T^*}{\pi} + \frac{q^* y}{k} - \sum_{m=1}^{\infty} \frac{4T^*}{(4m^2 - 1)\pi} \cos\left(\frac{2m\pi x}{L}\right) \exp\left(-\frac{2m\pi y}{L}\right).$$

   [*You may quote the formulae for the Fourier coefficients of a cosine series.*]

   (b) Hence, or otherwise, show that if y^* is a non-negative constant then

$$\int_0^L k \frac{\partial T}{\partial y}(x, y^*)\, \mathrm{d}x = Lq^*.$$

   What is the physical significance of this equality?

2. An infinite straight metal rod of constant thermal conductivity k has cross-section that is a path-connected region S bounded by a simple closed curve C. The temperature $T(x, y)$ in each cross-section satisfies Poisson's equation

$$-k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = Q(x, y) \quad \text{for} \quad (x, y) \in S,$$

with Newton's law of cooling giving the boundary condition

$$-k\frac{\partial T}{\partial n} = h\,(T - T_a) \quad \text{for} \quad (x, y) \in C,$$

where Q is the given volumetric heat source, h is the constant heat transfer coefficient, T_a is the constant ambient temperature and $\partial T / \partial n$ denotes the outward normal derivative of T on C.

   (a) Use *Green's Theorem* to show that there is at most one solution if $h > 0$, and that if $h = 0$ then any two solutions differ by a constant.

   (b) Now take the region S to be the disc of radius a with centre at the origin $(0, 0)$.

      (i) Find the cylindrically symmetric solution if Q is constant and $h > 0$.

      (ii) Give an example to show that there exists a $h < 0$ for which the solution is not unique if $Q = 0$ and $T_a = 0$.

3. (a) Suppose that the temperature $T(r,\theta)$ satisfies Laplace's equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0 \quad \text{for} \quad r > 0,$$

where (r,θ) are plane polar coordinates. Use the method of separation of variables and the principle of superposition to derive the general series solution given by

$$T(r,\theta) = A_0 + B_0 \log(r) + \sum_{n=1}^{\infty} \left(\left(A_n r^n + \frac{B_n}{r^n} \right) \cos(n\theta) + \left(C_n r^n + \frac{D_n}{r^n} \right) \sin(n\theta) \right),$$

where A_n, B_n, C_n and D_n are constants. You should state where you impose the constraint that the solution is periodic in θ with period 2π.

(b) Let a, b, T^* and q^* be positive constants, with $a < b$, and let k be the constant thermal conductivity. Derive the solution $T(r,\theta)$ of Laplace's equation in the region

(i) $r > a$ with the boundary conditions

$$T(a,\theta) = T^* \cos^2\theta \quad \text{and} \quad \lim_{r\to\infty} r\frac{\partial T}{\partial r}(r,\theta) = 0 \quad \text{for} \quad -\pi < \theta \le \pi;$$

(ii) $a < r < b$ with the boundary conditions

$$T(a,\theta) = T^* \cos(\theta) \quad \text{and} \quad -k\frac{\partial T}{\partial r}(b,\theta) = q^* \quad \text{for} \quad -\pi < \theta \le \pi;$$

(iii) $r < a$ with the boundary conditions

$$T(a,\theta) = \begin{cases} 0 & \text{for } -\pi < \theta \le 0, \\ T^* & \text{for } 0 < \theta \le \pi. \end{cases}$$

[*You may quote the formulae for the Fourier coefficients of a Fourier series.*]

(c) Find the heat flux $\mathbf{q}\cdot\mathbf{n}$ out of each region in part (b) through the boundary at $r = a$, where the heat flux vector $\mathbf{q} = -k\boldsymbol{\nabla}T$ according to Fourier's Law and in each case $\mathbf{n}$ is the outward pointing unit normal to $r = a$.

[*Before taking the limit $r \to a-$ in case (iii), you may find it helpful to sum the series solution for $T_r(r,\theta)$ by making a suitable choice for z in the identity*

$$\frac{z}{1-z^2} = \sum_{m=0}^{\infty} z^{2m+1},$$

which is valid for $z \in \mathbb{C}$ such that $|z| < 1$.]