MULTIVARIABLE CALCULUS HT20 SHEET 3
Surface integrals. Flux Integrals. Solid Angle.

1. Parameterize the surface of a cone, with base radius a and height h, and so express its surface area as a multiple integral. Hence show its surface area equals

$$\pi a\sqrt{a^2+h^2}.$$

Re-evaluate this surface area by considering how a sector of a disc may be folded to make a cone.

2. Show that the solid angle at the apex of a cone with semiangle α is $2\pi\left(1-\cos\alpha\right)$.

If a sphere has radius R and its centre at distance D from an observer, with $D \gg R$, show that the sphere occupies, as a fraction

$$\frac{1}{2}\left(1-\frac{\sqrt{D^2-R^2}}{D}\right)\approx\frac{R^2}{4D^2}$$

of the observer's view.

Use this to explain how the sun (at radius 7×10^5 km and distance 1.5×10^8 km) and moon (at radius 1.8×10^3 km and distance 3.8×10^5 km) occupy roughly the same amount of the sky.

3. Evaluate

$$\iint_{\Sigma}\mathbf{F}\cdot\mathrm{d}\mathbf{S}$$

where $\mathbf{F}=\left(\left(x-1\right)x^2y,\left(y-1\right)^2xy,z^2-1\right)$ and Σ is the surface of the unit cube $\left[0,1\right]^3$.

4. Two points are chosen at random on the surface of the sphere Σ with equation $x^2+y^2+z^2=a^2$. Explain why the mean distance μ between the points equals the integral

$$\mu=\frac{1}{4\pi a^2}\iint_{\Sigma}\sqrt{x^2+y^2+(z-a)^2}\,\mathrm{d}S$$

and hence determine μ.

5. (i) Calculate the surface integrals $\iint_\Sigma f\,\mathrm{d}S$ and $\iint_\Sigma f\,\mathrm{d}\mathbf{S}$ where

$$f(x,y,z)=(x^2+y^2+z^2)^2$$

and

$$\Sigma=\left\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=z^2,\ y\geqslant 0,\ 0\leqslant z\leqslant 2\right\}.$$

(ii) Parametrise the various parts of the boundary $\partial\Sigma$ and determine $\int_{\partial\Sigma} f\,\mathrm{d}s$ and $\int_{\partial\Sigma} f\,\mathrm{d}\mathbf{r}$.

6. (Optional) A spherical shell Σ with equation $x^2+y^2+z^2=1$ has density $\rho(x,y,z)\geqslant 0$. Show that its moment of inertia about an axis through the points $(\pm a,\pm b,\pm c)$ on the shell equals

$$I(a,b,c)=\iint_{\Sigma}(1-(ax+by+cz)^2)\rho(x,y,z)\,\mathrm{d}S.$$

Find this value when ρ is constant. Conversely, if $I(a,b,c)$ is constant for all $(a,b,c)\in\Sigma$, need $\rho(x,y,z)$ be constant?