MULTIVARIABLE CALCULUS HT20 SHEET 4
Div, Grad and Curl. Physical Interpretation. Identities.

1. Show directly that $\nabla^2(\phi\psi) = \phi\nabla^2\psi + 2\nabla\phi\cdot\nabla\psi + \psi\nabla^2\phi$ for scalar fields ϕ and ψ.

2. (i) Let $\phi(x,y,z) = y^2 - xz$ and $\mathbf{f}(x,y,z) = (z^2, x^2, y^2)$. Find $\nabla\phi$ and $\nabla\cdot\mathbf{f}$.

(ii) For the orthonormal basis $\mathbf{e}_1 = (0,-1,0)$, $\mathbf{e}_2 = (1,0,-1)/\sqrt{2}$, $\mathbf{e}_3 = (1,0,1)/\sqrt{2}$, create new co-ordinates X, Y, Z such that

$$X\mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3 = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Determine x, y, z in terms of X, Y, Z.

(iii) Find Φ, F_1, F_2, F_3 such that $\Phi(X,Y,Z) = \phi(x,y,z)$ and $F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3 = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$. Verify, by direct calculation, that

$$\nabla\phi = \Phi_X\mathbf{e}_1 + \Phi_Y\mathbf{e}_2 + \Phi_Z\mathbf{e}_3; \qquad \nabla\cdot\mathbf{f} = (F_1)_X + (F_2)_Y + (F_3)_Z.$$

3. Let r and θ denote plane polar co-ordinates and set $\mathbf{e}_r = (\cos\theta, \sin\theta, 0)$ and $\mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)$. Let $\mathbf{F}(r,\theta) = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta$ be a vector field. Prove that

$$\nabla\cdot\mathbf{F} = \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta}.$$

4. Let $\mathbf{f}(x,y,z) = (y/(x^2+y^2), -x/(x^2+y^2), 0)$ where $(x,y) \neq (0,0)$.

(i) Show that $\operatorname{curl}\mathbf{f} = \mathbf{0}$.

(ii) Find $\int_C \mathbf{f}\cdot d\mathbf{r}$ for each of the following closed curves C.

(a) C is parametrised by $\mathbf{r}(t) = (\cos t, \sin t, 0)$ for $0 \leqslant t \leqslant 2\pi$.

(b) C is parametrised by

$$\mathbf{r}(t) = \begin{cases} (\cos t, \sin t, t) & 0 \leqslant t \leqslant 4\pi, \\ (1, 0, 8\pi - t) & 4\pi \leqslant t \leqslant 8\pi. \end{cases}$$

(c) C is the square with vertices $(0,1), (1,1), (1,2), (0,2)$ with an anticlockwise orientation.

(iii) Find a scalar field ϕ such that $\mathbf{f} = \nabla\phi$ on $R_1 = \{(x,y,z) : y > 0\}$. How does the existence of ϕ relate to your answer to **4**(ii)(c)?

(iv) Show that there does not exist ψ such that $\mathbf{f} = \nabla\psi$ on $R_2 = \{(x,y,z) : (x,y) \neq (0,0)\}$.

5. (i) With $\mathbf{f}$ as in Exercise 4, show that $\operatorname{div}\mathbf{f} = 0$.

(ii) Suppose that a particle $(x(t), y(t))$ moves according to the flow

$$dx/dt = y/\left(x^2+y^2\right), \qquad dy/dt = -x/\left(x^2+y^2\right).$$

Show that, on changing to polar co-ordinates (r,θ) these differential equations become

$$dr/dt = 0, \qquad d\theta/dt = -1/r^2.$$

(iii) Suppose that particles initially occupy the region $R_0 = \{(r,\theta) : 0 < a < r < b,\ 0 < \theta < \alpha < \pi/2\}$. If the particles move according to the above flow, describe the region R_t which they occupy a short time t afterwards. Sketch R_0 and R_t, and show that the regions have the same area.

6. (Optional) (i) In spherical polar co-ordinates

$$\mathbf{r}(r, \theta, \phi) = (r \sin\theta \cos\phi, r \sin\theta \sin\phi, r \cos\theta).$$

Show that we can write

$$\mathrm{d}\mathbf{r} = h_r \mathrm{d}r \mathbf{e}_r + h_\theta \mathrm{d}\theta \mathbf{e}_\theta + h_\phi \mathrm{d}\phi \mathbf{e}_\phi$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are a right-handed orthonormal basis and $h_r, h_\theta, h_\phi > 0$. Show that $h_r h_\theta h_\phi = r^2 \sin\theta = \partial(x, y, z)/\partial(r, \theta, \phi)$.

(ii) More generally $\mathbf{r}(u_1, u_2, u_3)$ is a parametrization of space by *orthogonal curvilinear co-ordinates* if

$$\mathrm{d}\mathbf{r} = h_1 \mathrm{d}u_1 \mathbf{e}_1 + h_2 \mathrm{d}u_2 \mathbf{e}_2 + h_3 \mathrm{d}u_3 \mathbf{e}_3$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are a right-handed orthonormal basis. Show in this case that

$$\frac{\partial(x, y, z)}{\partial(u_1, u_2, u_3)} = h_1 h_2 h_3.$$

(iii) Show further for a scalar field $\Phi(u_1, u_2, u_3)$ that

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\mathbf{e}_3.$$