MULTIVARIABLE CALCULUS HT20 SHEET 5
Green's theorems. Divergence theorem.

1. (i) Let $\mathbf{F}(x,y,z) = (3x^2y^2z, 2x^3yz, x^3y^2)$. Show that $\nabla \wedge \mathbf{F} = \mathbf{0}$ and find a potential ϕ such that $\mathbf{F} = \nabla\phi$. To what extent is ϕ unique?

Verify by direct calculation that

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \phi(\mathbf{q}) - \phi(\mathbf{p})$$

where $\mathbf{p} = (0,0,0)$, $\mathbf{q} = (1,1,1)$ and C is the *twisted cubic* $\mathbf{r}(t) = (t, t^2, t^3)$ with $0 \leqslant t \leqslant 1$.

(ii) Let $\mathbf{F}(x,y,z) = (0, xy-1, y-xz)$. Show that $\nabla \cdot \mathbf{F} = 0$ and that $\mathbf{f}(x,y,z) = (xyz, xy, x)$ is a vector potential – that is $\mathbf{F} = \nabla \wedge \mathbf{f}$. To what extent is $\mathbf{f}$ unique?

2. Use Green's theorem to find the simple closed curve C in the xy-plane that maximises the integral

$$\int_C y^3\,\mathrm{d}x + \left(3x - x^3\right)\,\mathrm{d}y,$$

and determine this maximum.

3. Let R be a closed bounded region, bounded by a closed surface ∂R. Let ϕ, ψ be smooth scalar fields on R. Use the divergence theorem to prove:

(i) **Green's First Theorem:**

$$\iiint_R \left(\psi\nabla^2\phi + \nabla\phi \cdot \nabla\psi\right)\mathrm{d}R = \iint_{\partial R} \psi\nabla\phi \cdot \mathrm{d}\mathbf{S}.$$

(ii) **Green's Second Theorem:**

$$\iiint_R \left(\psi\nabla^2\phi - \phi\nabla^2\psi\right)\mathrm{d}R = \iint_{\partial R} \left(\psi\nabla\phi - \phi\nabla\psi\right) \cdot \mathrm{d}\mathbf{S}.$$

4. Verify the divergence theorem where $\mathbf{F}(x,y,z) = (y, xy, -z)$, and R is the region enclosed below the plane $z = 4$,and the paraboloid $z = x^2 + y^2$.

5. Let f be a smooth scalar field defined on a region $R \subseteq \mathbb{R}^3$ with a smooth boundary ∂R. Show that

$$\iint_{\partial R} f\mathbf{r} \wedge \mathrm{d}\mathbf{S} = \iiint_R \mathbf{r} \wedge \nabla f\,\mathrm{d}V.$$

6. (Optional) A one-dimensional blob of compressible fluid starts at $t = 0$ with uniform density $\rho = 1$ in the interval $1 \leqslant x \leqslant 2$ and moves with velocity given by

$$u(x,t) = 2x^2t.$$

(i) Find the position $x(t)$ at time t of a fluid particle that starts at $x(0) = a$ where $1 \leqslant a \leqslant 2$.

(ii) Hence show that the density of the fluid is given by

$$\rho(x,t) = \frac{1}{(1+xt^2)^2}.$$

(iii) Verify that $\rho(x,t)$ satisfies the *continuity equation*

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

(iv) Use the continuity equation to find a condition on $u(x,t)$ for a fluid to be incompressible.