MULTIVARIABLE CALCULUS HT20 SHEET 7
Stokes' theorem. Examples. Consequences.

1. Let $0 < a < b$. Verify Stokes' Theorem when $\mathbf{F} = (y, z, x)$ and Σ is the upper half of the torus generated by rotating the circle $(x-b)^2 + z^2 = a^2$ about the z-axis.

2. The vector field $\mathbf{F}(\mathbf{R})$ is defined by

$$\mathbf{F}(\mathbf{R}) = \int_C |\mathbf{r} - \mathbf{R}|^2 \, \mathrm{d}\mathbf{r}$$

where $\mathbf{r}$ lies on the simple closed curve C. Show that there are constant vectors $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{F}(\mathbf{R}) = \mathbf{R} \wedge \mathbf{A} + \mathbf{B}$. Deduce that

$$\nabla \wedge \mathbf{F} = -4 \iint_S \mathrm{d}\mathbf{S}$$

where S is any smooth surface spanning C.

3. Let Σ denote that part of the cone $x^2 + y^2 = z^2$, $z > 0$ which lies beneath the plane $x + 2z = 1$. Let $\mathbf{F}(x, y, z) = x\mathbf{j}$.

Show that the projection of $\partial\Sigma$ vertically to the xy-plane is an ellipse. Parametrise $\partial\Sigma$ and determine $\int_{\partial\Sigma} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$.

Show that $\mathrm{d}\mathbf{S} \cdot \mathbf{k} = \mathrm{d}x\,\mathrm{d}y$ on Σ and verify Stokes' Theorem for $\mathbf{F}$ on Σ.

4. Let $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ be defined on $\mathbb{R}^2$ where u, v have continuous partial derivatives of all orders.

(i) Under what condition on u and v is $\operatorname{curl}\mathbf{F} = \mathbf{0}$? Under what condition is $\operatorname{div}\mathbf{F} = 0$? Show that if both these conditions hold then u and v are harmonic – that is, they satisfy Laplace's equation.

(ii) Conversely say U is a harmonic function. Use Green's Theorem to explain why

$$\mathbf{G}(X, Y) = \left(U(X, Y), \int_C U_y \, \mathrm{d}x - U_x \, \mathrm{d}y \right)$$

is a well-defined function, independent of the choice of curve C from $(0, 0)$ to (X, Y). Show that $\operatorname{div}\mathbf{G} = 0$ and $\operatorname{curl}\mathbf{G} = \mathbf{0}$.

(iii) What is $\mathbf{G}(x, y)$ if (a) $U(x, y) = x^2 - y^2$? (b) $U(x, y) = e^x \cos y$?

5. (Optional) Let $\mathbf{F} = \mathbf{r}/r^3 = \mathbf{e}_r/r^2$, in terms of spherical polar co-ordiantes, and let

$$R_1 = \mathbb{R}^3 \backslash \{\mathbf{0}\} \qquad \text{and} \qquad R_2 = \left\{ (x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 \neq 0 \right\}.$$

(i) Show that $\operatorname{div}\mathbf{F} = 0$.

(ii) Show that

$$\mathbf{f} = \frac{\cot\theta}{r} e_\phi$$

is a vector potential for $\mathbf{F}$ on R_2 – that is, show that $\mathbf{F} = \nabla \wedge \mathbf{f}$.

(iii) Why is $\mathbf{f}$ not a vector potential for $\mathbf{F}$ on R_1?