**MULTIVARIABLE CALCULUS HT20 SHEET 8**
**Gravity. Gauss' Flux Theorem. Poisson's Equation.**

**1.** Two planets, modelled as particles of equal mass $M$, are situated at the points $(a, 0, 0)$ and $(-a, 0, 0)$ and remain there. Show that the gravitational potential at the point $(x, 0, 0)$, where $-a < x < a$, is

$$\frac{2GMa}{a^2 - x^2},$$

and find the gravitational field at $(x, 0, 0)$. A space-craft moves along the segment $-a < x < a$ of the $x$-axis under the gravitational attraction of the planets. Write down the equation of motion for the space-craft and show that $x = 0$ is a point of unstable equilibrium.

At $x = 0$ the space-craft fires its motor briefly so as to acquire a speed $u(> 0)$ and heads towards $x = a$. By using energy considerations show that the time taken to reach $x = a$ is

$$T = \int_0^a \sqrt{\frac{a^2 - x^2}{u^2(a^2 - x^2) + \frac{4GM}{a}x^2}}\, \mathrm{d}x.$$

(You are not required to evaluate the integral.)

**2.** A homogeneous straight wire of mass $M$ lies along the $x$-axis from $(-a, 0, 0)$ to $(a, 0, 0)$. Show that the gravitational potential at the point $(x, y, 0)$, where $y \neq 0$, is

$$\frac{GM}{2a}\left[\sinh^{-1}\left(\frac{a - x}{|y|}\right) + \sinh^{-1}\left(\frac{a + x}{|y|}\right)\right]$$

and that the $x$-component of the gravitational field is

$$\frac{GM}{2a}\left(\frac{1}{d_1} - \frac{1}{d_2}\right),$$

where $d_1$ and $d_2$ are the distances from $(x, y, 0)$ to the ends of the wire.

**3.** Show that the gravitational field at the vertex of a homogeneous circular cone is

$$\frac{12GM}{a^2}\sin^2\frac{\alpha}{2},$$

where $M$ is the mass of the cone, $a$ is the radius of the base and $\alpha$ is the semi-vertical angle.

**4.** A hollow spherical shell has internal radius $a$ and external radius $b$, and is made of material of uniform density $\rho$. Find the gravitational field and the gravitational potential in the three regions $0 < r < a,\ a < r < b,\ r > b$ by using (1) the Flux Theorem, and (2) Poisson's equation.

**5.** A solid hemisphere of uniform density $\rho$ occupies the region

$$x^2 + y^2 + z^2 \leqslant a^2, \qquad z \leqslant 0.$$

Find the gravitational potential due to the hemisphere at the point $(0, 0, s)$ where $s > 0$. A uniform rod of density $m$ per unit length lies on the $z$-axis between $(0, 0, c)$ and $(0, 0, d)$ where $d > c > 0$. Show that the force exerted on the rod by the hemisphere is

$$\psi(c) - \psi(d)$$

where

$$\psi(\lambda) = \frac{2\pi Gm\rho}{3}\left(\frac{a^3 + \lambda^3 - \left(a^2 + \lambda^2\right)^{3/2}}{\lambda}\right).$$