# Dynamics

James Sparks, Hilary Term 2020

### About these notes

These are lecture notes for the Prelims Dynamics course, which is a first year course in the mathematics syllabus at the University of Oxford. In putting together the notes I have drawn freely from the enormous literature on the subject; most notably from previous lecture notes for this course (due to David Acheson and Jon Chapman) and the reading list, but also from many other books and lecture notes. The notes are unchanged from last year.

Oxford mathematics students studying Dynamics will have taken a first course in geometry, covering the elementary ideas of the geometry of Euclidean space, including invariance under orthogonal transformations, and a first course in calculus, in particular covering simple ordinary differential equations. Some familiarity with these topics will hence be assumed. Starred sections/paragraphs are not examinable, either because the material is slightly off-syllabus, or because it is more difficult. There are eight (short) problem sheets. Please send any questions/corrections/comments to `sparks@maths.ox.ac.uk`.[1]

## Contents



[1] Many thanks to Pietro Benetti Genolini and Agniete Geras for comments on the first version of these notes.

## Preamble

Newtonian mechanics, as first developed by Galileo and Newton in the 17th century, is an extraordinarily successful theory. Its laws are clear and relatively simple to state, but are applicable to an enormous array of dynamical problems. They are also valid over a vast range of scales. For example, in these lectures we'll see that Newton's laws govern phenomena as diverse as the motion of bodies through fluids, charged particles moving in electromagnetic fields, the motion of rigid bodies under gravity, and perhaps most famously the orbits of planets in our solar system. There are also the slightly more mundane examples: masses attached to springs and rods, marbles rolling on surfaces, beads sliding on wires, *etc.* For applied mathematicians the ideas and techniques developed in Newtonian mechanics have wide applicability, from phenomena in dynamical systems, such as resonance and chaos, to *e.g.* the mathematical modelling of biological systems.

Newton's laws nevertheless have their limits. For physics at the atomic scale classical mechanics is replaced by quantum mechanics, while for phenomena involving speeds approaching the speed of light one needs Einstein's theory of relativity. However, these are much more complex descriptions of Nature. Since for scales of everyday experience these theories agree with Newtonian mechanics, to a good approximation, they are simply not needed to accurately describe many phenomena. In quantum mechanics and relativity many concepts in Newton's theory are modified: the concepts of space and time, the notion of a particle trajectory, and even the basic process of measurement, are all radically altered. Nevertheless, many features of Newtonian mechanics appear to be fundamental. In particular, the laws of conservation of energy, momentum and angular momentum developed in this course are in some sense universal, and pervade all of theoretical physics.

# 1 Newtonian mechanics

## 1.1 Space and time

In Newtonian mechanics space is described by Euclidean geometry. In order to make this precise we introduce the notion of a *reference frame*.

**Definition** A *reference frame* $\mathcal{S}$ is specified by a choice of origin $O$, together with a set of perpendicular (right handed) Cartesian coordinate axes at $O$.

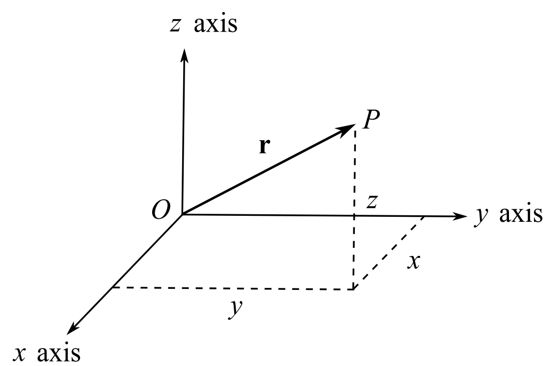



Figure 1: The position vector $\mathbf{r} = (x, y, z)$ of a point $P$, as measured in a reference frame $\mathcal{S}$.

With respect to $\mathcal{S}$ a point $P$ is specified by a position vector $\mathbf{r}$ from $O$ to $P$. The chosen Cartesian coordinate axes allow us to write $\mathbf{r}$ in terms of its components $\mathbf{r} = (x, y, z)$. The Euclidean distance between two points $P_1, P_2$ with position vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$, $\mathbf{r}_2 = (x_2, y_2, z_2)$ is $|\mathbf{r}_1 - \mathbf{r}_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

For many problems there may be a natural or convenient choice of reference frame, although this is not always the case. An important assumption in Newtonian mechanics is that any two observers, using any choice of reference frames, agree on their measurements of distances – provided they use the same units, which we take to be metres m. If we fix an initial choice of $\mathcal{S}$, then the origin $O'$ of any other reference frame $\mathcal{S}'$ will be at some position $\mathbf{x}$, measured from the origin $O$ of $\mathcal{S}$. See Figure 2. In order that distances measured in the two frames are the same, the coordinate axes of $\mathcal{S}'$ must differ from those of $\mathcal{S}$ by a $3 \times 3$ rotation, *i.e.* an orthogonal transformation.[2] At some level these statements might seem intuitively obvious, but they were formalised in the Geometry course last term: the two reference frames both identify space with Euclidean $\mathbb{R}^3$, and you proved that any distance-preserving map (an *isometry*) between the two is necessarily a combination of a translation and orthogonal transformation. Thus if $\mathbf{r} = (x, y, z)$ denotes the position of a point $P$ in the frame $\mathcal{S}$, and $\mathbf{r}' = (x', y', z')$ is the position of the same point in the frame $\mathcal{S}'$, we have

$$\mathbf{r}' \quad = \quad \mathcal{R}\,(\mathbf{r} - \mathbf{x}) \ , \tag{1.1}$$

[2] A general $3 \times 3$ orthogonal transformation is either a rotation, a reflection, or a combination of a rotation and reflection, but a single reflection takes a right handed frame to a left handed frame.

where $\mathcal{R}$ is a $3 \times 3$ orthogonal matrix. Recall these are characterized by $\mathcal{R}^{\mathrm{T}} = \mathcal{R}^{-1}$.

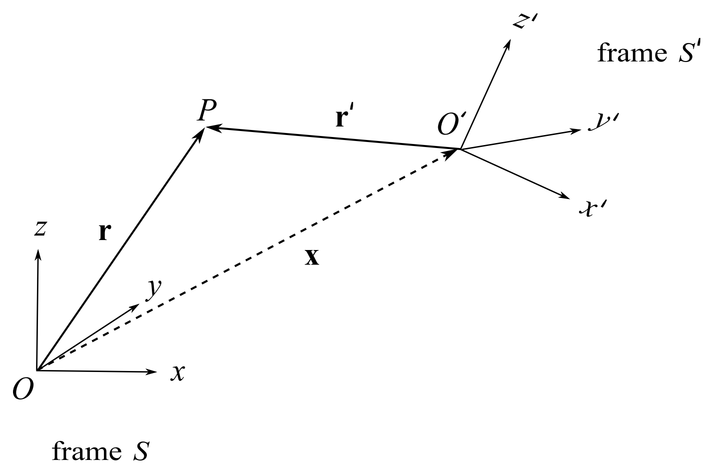



Figure 2: Relative to a choice of reference frame $\mathcal{S}$, the origin $O'$ of another reference frame $\mathcal{S}'$ has position vector $\mathbf{x}$, and the coordinate axes of $\mathcal{S}'$ differ from those of $\mathcal{S}$ by a rotation.

In order to describe dynamics we also need time. In Newtonian mechanics there is a notion of *absolute time*: provided any two observers use the same units of time, which we take to be seconds s, they will always agree on the time interval between any two events. This means that the time variables used by two different observers are related by $t' = t - t_0$, and they are always free to synchronize their clocks to set $t_0 = 0$.

Returning to the two reference frames in Figure 2, the origins $O$, $O'$ may move relative to each other, so $\mathbf{x} = \mathbf{x}(t)$, and the axes may also rotate, so the orthogonal transformation $\mathcal{R} = \mathcal{R}(t)$ is time-dependent. We shall describe rotating frames in much greater detail in section 8.

### 1.2 Newton's laws

Many dynamical processes in the real world are clearly very complicated. Mathematical models of dynamical systems usually involve making various approximations, or idealizations, in the description of the system. One usually wants to construct the simplest model that captures the most important features of the dynamics. Most of this course will focus on the dynamics of *point particles*. These are objects whose dimensions may be neglected, to a good approximation, in describing their motion. For example, this is the case if the size of the object is small compared to the distances involved in the dynamics; *e.g.* the motion of the Earth around the Sun may be described very accurately by treating the Earth and Sun as point particles. On the other hand, it's no good treating the Earth as a point particle if you want to understand the effects of its rotation!

**Definition** A *point particle* is an idealized object that at a given instant of time $t$ is located at a point $\mathbf{r}(t)$, as measured in some reference frame $\mathcal{S}$. The *velocity* of the particle is $\mathbf{v} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$, where a dot will denote derivative with respect to time. Its *acceleration* is $\mathbf{a} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$.

**Example** (Motion with constant acceleration): Consider a particle moving in a straight line with constant acceleration $\mathbf{a}$. Let us orient our axes so that $\mathbf{a} = a\,\mathbf{k}$, where $\mathbf{k}$ is a unit vector in the increasing $z$ direction. Suppose that the particle starts at time $t = 0$ at the origin and has initial velocity $\mathbf{u} = u\,\mathbf{k}$.

The constant acceleration condition is a second order differential equation for $\mathbf{r}(t)$, namely $\ddot{\mathbf{r}} = a\,\mathbf{k}$. In Cartesian coordinates this reads $(\ddot{x}, \ddot{y}, \ddot{z}) = (0, 0, a)$. Integrating this equation once with respect to time $t$ gives

$$\dot{\mathbf{r}} \;=\; a\,t\,\mathbf{k} + \mathbf{c}\,, \tag{1.2}$$

where $\mathbf{c}$ is a vector integration constant. The initial condition that $\dot{\mathbf{r}}(0) = \mathbf{u} = u\,\mathbf{k}$ then determines $\mathbf{c} = u\,\mathbf{k}$. Integrating (1.2) again with respect to time $t$ gives the solution

$$\mathbf{r}(t) \;=\; \left(\tfrac{1}{2}a\,t^2 + u\,t\right)\,\mathbf{k} \;=\; (0, 0, \tfrac{1}{2}a\,t^2 + u\,t)\,. \tag{1.3}$$

Here we have used the initial condition that the particle starts at time $t = 0$ at the origin, so $\mathbf{r}(0) = \mathbf{0}$, to determine the second vector integration constant. ■

As time evolves the position of the particle sweeps out a curve $\mathbf{r}(t)$, parametrized by time $t$, which we refer to as the *trajectory*. This must satisfy Newton's laws of motion for point particles, but before discussing these we need another definition.

**Definition** A point particle has a (*inertial*) *mass* $m > 0$. We measure mass in kilograms kg. Its *momentum* (or more accurately *linear momentum*) is $\mathbf{p} = m\mathbf{v} = m\dot{\mathbf{r}}$.

In section 1.1 we noted that there are many choices of reference frames. Newton's first law singles out a special class of reference frames, called *inertial frames*.

> **N1**: In an *inertial frame* a particle moves with constant momentum, unless acted on by an external force.

In this course we will only consider constant mass particles, so that constant momentum $\mathbf{p} = m\mathbf{v}$ means *constant velocity* $\mathbf{v}$. This is also sometimes referred to as *uniform motion in a straight line*.

Suppose I choose a reference frame $\mathcal{S}$: how do I know it is inertial? According to **N1** it is inertial if a particle with no identifiable forces acting on it travels in a straight line with constant speed $v = |\mathbf{v}|$. But how do we know whether or not there are any forces acting? And indeed, what is a force?! We will begin to introduce and study forces in section 2, but an essential point is that forces arise from the presence of other matter, which our particle interacts with. Thus one way to ensure there are no forces acting is to head deep into space, far away from any other matter. This is not very practical. On the surface of the Earth every particle experiences the force of *gravity*. However, for a particle sitting on a solid surface the force due to gravity (its *weight*) is balanced by a normal reaction force of the surface pushing back on the particle. There is hence no *net* force acting on the particle, and the fact that it doesn't move demonstrates that a frame rigidly fixed relative to the surface of the Earth is a very good approximation to an inertial frame.[3] Whenever we refer to an "inertial frame", we usually have in mind such a frame fixed to the Earth's surface.

What about *non-inertial* frames? We shall describe these in much more detail in section 8, but it might be helpful here to make a few, hopefully intuitive, comments. Relative to an inertial frame $\mathcal{S}$, a non-inertial frame $\mathcal{S}'$ will either have: (i) the origin $O'$ accelerating with respect to $O$, or (ii) the axes of $\mathcal{S}'$ rotating relative to the axes of $\mathcal{S}$. In a non-inertial frame a particle will appear to be acted on by "fictitious forces", in addition to any actual forces in Newton's second law stated below. For example, consider an observer standing inside a train carriage, with reference frame $\mathcal{S}'$ fixed relative to the interior of the train. As the train pulls out of a station it accelerates, and the origin $O'$ of $\mathcal{S}'$ is likewise accelerating. The person inside the train (and everything else!) feels like they are being thrown backwards: this isn't a real force in Newton's equations, but a fictitious force due to the frame $\mathcal{S}'$ being non-inertial. Similarly, consider an observer standing on a roundabout, whose frame $\mathcal{S}'$ rotates with the roundabout about a fixed vertical axis. As most of us will have experienced, you feel like you are being thrown outwards, away from the axis of rotation.

In an *inertial frame*, the dynamics of a point particle is governed by

> **N2**: The rate of change of linear momentum is equal to the net force acting on the particle: $\mathbf{F} = \dot{\mathbf{p}}$.

Assuming the mass $m$ is constant the right hand side of Newton's second law is $\dot{\mathbf{p}} = m\ddot{\mathbf{r}}$, and this is the vector form of the familiar "$F = ma$". The inertial mass $m$ of a particle hence measures its resistance to accelerate when subjected to a given force $\mathbf{F}$. This external force might in general depend on the particle's position $\mathbf{r}$, its velocity $\dot{\mathbf{r}}$, and on time $t$, so that $\mathbf{F} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$. Newton's second law is then a second order ordinary differential equation (ODE) for $\mathbf{r}(t)$:

$$\mathbf{F}(\mathbf{r}(t), \dot{\mathbf{r}}(t), t) \quad = \quad m\,\ddot{\mathbf{r}}(t) \; . \tag{1.4}$$

This is also often referred to as the *equation of motion* for the particle. Since (1.4) is second order, for "suitably nice" functions $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$ one expects that specifying the position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ at some initial time $t = t_0$ gives a unique solution for the particle trajectory $\mathbf{r}(t)$. A central problem in Dynamics is to find this trajectory, for a given force $\mathbf{F}$.

Finally, if we have more than one particle, then

> **N3**: If particle 1 exerts a force $\mathbf{F} = \mathbf{F}_{21}$ on particle 2, then particle 2 also exerts a force $\mathbf{F}_{12} = -\mathbf{F}$ on particle 1.

[3] Actually it is not quite inertial: the Earth rotates around its axis once per day, and is accelerating due to its motion around the Sun once per year. The former leads to a measurable effect, as we shall see in section 8.5.

In other words, $\mathbf{F}_{12} = -\mathbf{F}_{21}$. This is often paraphrased by saying that every action has an equal and opposite reaction.

## 1.3 Dimensional analysis

The fundamental dimensions in mechanics are length L, time T and mass M.[4] A square bracket is usually used to denote the dimension of a variable, so that [length] = L, [time] = T, [mass] = M. Dimensions of other quantities may then be derived from these. For example, the dimensions of velocity are $[\dot{\mathbf{r}}] = \mathrm{L\,T}^{-1}$.

A given dimension may be measured in a number of different standard *units*. For example, length might be measured in inches, metres or light-years (the distance light travels in a year in vacuum). There is then a scaling factor to convert between different units, *e.g.* 1 metre $\simeq$ 39.4 inches, 1 light-year $\simeq 9.46 \times 10^{15}$ metres, *etc.* In order that equations in physics are independent of the choice of units, which after all are arbitrary, it's important that the dimensions of both sides of an equation are the same. Similarly, we may only add two quantities if they have the same dimensions.

**Example** (Dimensions of force): Newton's second law gives the dimensions of force as $[\mathbf{F}] = \mathrm{M\,L\,T}^{-2}$. The magnitude $|\mathbf{F}|$ is measured in *Newtons* N, where 1 N = 1 $\mathrm{kg\,m\,s}^{-2}$. ■

More interestingly, a knowledge of the dimensions of the parameters in a problem can sometimes be used to construct scaling laws, without needing to solve any differential equations.

**Example** (Maximum height for constant acceleration): Let's reconsider the example of constant acceleration in section 1.2. For a particle moving along the $z$ axis, starting at the origin at time $t = 0$ with velocity $\mathbf{u} = u\,\mathbf{k}$, we showed that the trajectory is $\mathbf{r}(t) = (\frac{1}{2}at^2 + ut)\,\mathbf{k}$. Suppose that $u > 0$ but the constant acceleration $a = -g < 0$ is negative; that is, the particle starts out moving in the positive $z$ direction, but is accelerating in the opposite direction. In this case it will reach a maximum height $z_{\mathrm{max}}$ at a time $t_{\mathrm{max}}$, when $\dot{\mathbf{r}}(t_{\mathrm{max}}) = \mathbf{0}$:

$$\mathbf{0} \;=\; \dot{\mathbf{r}}(t_{\mathrm{max}}) \;=\; (-g\,t_{\mathrm{max}} + u)\,\mathbf{k} \qquad \Longrightarrow \qquad t_{\mathrm{max}} \;=\; \frac{u}{g}\;. \tag{1.5}$$

We then compute

$$z_{\mathrm{max}} \;=\; -\frac{1}{2}g\,t_{\mathrm{max}}^2 + u\,t_{\mathrm{max}} \;=\; \frac{u^2}{2g}\;. \tag{1.6}$$

The dimensionful quantities in the problem are $u$, with $[u] = \mathrm{L\,T}^{-1}$, and $g$, with $[g] = \mathrm{L\,T}^{-2}$. The only way to obtain quantities with dimensions of T and L, respectively, are hence as

$$\left[\frac{u}{g}\right] \;=\; \frac{\mathrm{L\,T}^{-1}}{\mathrm{L\,T}^{-2}} \;=\; \mathrm{T}\;, \qquad \left[\frac{u^2}{g}\right] \;=\; \frac{\mathrm{L}^2\,\mathrm{T}^{-2}}{\mathrm{L\,T}^{-2}} \;=\; \mathrm{L}\;. \tag{1.7}$$

[4] When we discuss problems in electromagnetism we will also need to add electric charge Q.

Dimensional analysis thus tells us that $t_{\rm max}$ must be a dimensionless number times $u/g$, while $z_{\rm max}$ must be a dimensionless number times $u^2/g$. ■

An important role is played by *dimensionless* combinations of parameters in a problem. One reason for this is that only dimensionless parameters can appear as arguments in many of the functions that arise as solutions to differential equations, such as $\mathrm{e}^x$, $\sin x$, *etc.* To see this, note that *e.g.* the exponential function is defined as a power series $\mathrm{e}^x = 1 + x + \frac{1}{2!}x^2 + \cdots$, and so the variable $x$ must be dimensionless. For this reason, the same dimensionless combinations of parameters often appear again and again when solving a problem: it can be useful to recognize this, and rename these variables to simplify notation. Another comment is that dimensionless quantities can be large or small, while dimensionful quantities always have to be large or small *compared to another quantity with the same dimensions*. For example, is 1 metre (a dimensionful quantity) large or small? It's extremely large compared to the diameter of a hydrogen atom (approximately $10^{-10}$ m) but extremely small compared to the diameter of the observable universe (approximately $10^{27}$ m)! As another example, a dynamical system might have a dimensionless parameter $Q$, with qualitatively different behaviour for $Q > 1$ and $Q < 1$, with a critical behaviour for $Q = 1$. The dynamics might also simplify in the limit where certain dimensionless parameters become large (say $Q \to \infty$) or small ($Q \to 0$), allowing one to find analytic solutions to the equations in these limits.

# 2 Forces and dynamics: a first look

In this section we introduce a number of different forces, and solve Newton's second law (1.4) to find the particle trajectory $\mathbf{r}(t)$. In some cases more than one force may be acting on the particle. Forces are *vectors*, and the *total force* acting is simply the sum of all forces. Explicitly, if forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ all act on a particle, the force $\mathbf{F}$ appearing in Newton's second law is the *vector sum*

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i . \tag{2.1}$$

## 2.1 Gravity and projectiles

A particle of mass $m$ near the Earth's surface experiences a gravitational force $mg$ vertically downwards, where $g \simeq 9.81\ \mathrm{m\,s^{-2}}$ is the *acceleration due to gravity*. This force is the particle's *weight*. In an inertial frame where the $z$ axis is the vertical direction, so that the $x$ and $y$ axes are horizontal, we may write the force as $\mathbf{F} = -mg\,\mathbf{k}$, where $\mathbf{k}$ is a unit vector directed upwards. More precisely, the mass $m = m_G$ that appears in this force is the *gravitational mass*, which is logically distinct from the *inertial mass* $m = m_I$ that appears in **N2**. Newton's second law (1.4) hence reads

$$-m_G\, g\,\mathbf{k} = m_I\,\mathbf{a} . \tag{2.2}$$

It is an experimental fact that $m_I = m_G$, as demonstrated famously by Galileo throwing things off the tower of Pisa. It follows that the acceleration $\mathbf{a} = -g\,\mathbf{k}$ is independent of the mass (hence the name "acceleration due to gravity" for $g$). In practice air resistance can make an enormous difference when you throw two objects of the same mass, but more modern experiments confirm that $m_I/m_G = 1$, to at least $10^{-12}$ in precision.[5]

**Example** (Vertical motion under gravity): With notation as above, consider a particle of mass $m$ projected from the origin at time $t = 0$ with initial velocity $\mathbf{u} = u\,\mathbf{k}$. Newton's second law (2.2) simplifies to $\ddot{\mathbf{r}} = \mathbf{a} = -g\,\mathbf{k}$, which is precisely the example we solved in section 1. The solution is

$$\mathbf{r}(t) = \left(-\tfrac{1}{2}gt^2 + ut\right)\mathbf{k} . \tag{2.3}$$

■

We may make this more interesting by changing the initial condition.

**Example** (Projectiles): Suppose that a small projectile is thrown with velocity $\mathbf{V}$ at an angle $\alpha$ to the horizontal, from a height $h$ above the ground. Find the curve traced out by the trajectory of the projectile, and its horizontal range.

[5]** Einstein turned this around and made $m_I = m_G$ into a new principle, called the *Equivalence Principle*. It led him to formulate his General Theory of Relativity, in which gravity is not a force as in Newton's theory, but rather a curvature of space (which is no longer Euclidean) and time itself.

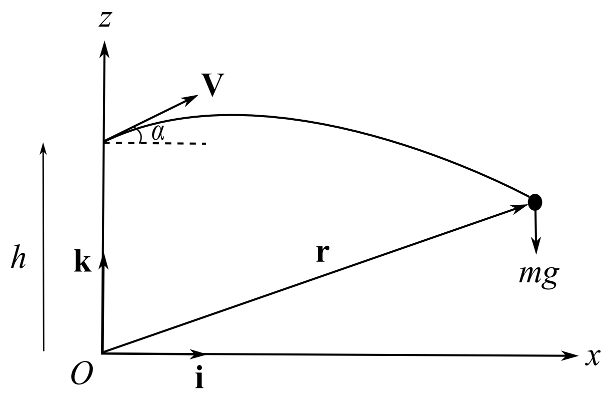



Figure 3: Throwing a projectile.

We choose the origin $O$ at ground level, and a unit vector $\mathbf{k}$ pointing vertically, and $\mathbf{i}$ horizontally along the ground. The only force acting is gravity, with $\mathbf{F} = -mg\,\mathbf{k}$, so that Newton's second law reads

$$m\ddot{\mathbf{r}} \;=\; -mg\,\mathbf{k}\ . \tag{2.4}$$

The initial conditions are

$$\text{At time } t=0: \qquad \mathbf{r}(0) \;=\; h\,\mathbf{k}\ , \qquad \dot{\mathbf{r}}(0) \;=\; \mathbf{V} \;=\; V\cos\alpha\,\mathbf{i} + V\sin\alpha\,\mathbf{k}\ . \tag{2.5}$$

Integrating (2.4) twice and using (2.5) we find the solution

$$\mathbf{r}(t) \;=\; -\frac{1}{2}g\,t^2\,\mathbf{k} + tV\cos\alpha\,\mathbf{i} + tV\sin\alpha\,\mathbf{k} + h\,\mathbf{k}\ . \tag{2.6}$$

This is the trajectory of the projectile. We can find the curve that this traces out in the $(x, z)$ plane by eliminating time $t$. Writing $\mathbf{r} = x\,\mathbf{i} + z\,\mathbf{k}$, reading off the components of (2.6) gives

$$x(t) \;=\; tV\cos\alpha\ , \qquad z(t) \;=\; -\frac{1}{2}g\,t^2 + tV\sin\alpha + h\ . \tag{2.7}$$

Using the first equation we may solve for $t$ in terms of $x$, and then substitute into the second equation, giving

$$z \;=\; -\frac{g}{2V^2}x^2\sec^2\alpha + x\tan\alpha + h\ . \tag{2.8}$$

This is a *parabola*.

The projectile hits the ground when $z = \mathbf{r}\cdot\mathbf{k} = 0$. From (2.8) this gives a *quadratic* equation for the horizontal range $x$, with solution

$$x \;=\; \frac{V^2\cos\alpha}{g}\left[\sin\alpha + \sqrt{\sin^2\alpha + 2gh/V^2}\right]\ . \tag{2.9}$$

Notice that the second solution to the quadratic, with a minus sign in front of the square root in (2.9), has $x < 0$ and corresponds to continuing the trajectory backwards, before $t = 0$. Note also that if we throw the projectile from ground level, so $h = 0$, the range simplifies to $x = (2V^2\cos\alpha\sin\alpha)/g = (V^2\sin 2\alpha)/g$, which is maximized to $x_{\mathrm{max}} = V^2/g$ for an angle $\alpha = \pi/4$. ■

## 2.2 Fluid drag

In practice any body moving through a fluid, such as air or water, experiences an effective *drag force*. This drag force is velocity dependent, with two common models being linear or quadratic in the speed, with the force acting in the opposite direction to the velocity of the particle:

- A linear drag holds when viscous forces predominate, *i.e.* this is due to the "stickiness" of the fluid. The force is

$$\mathbf{F} = -b\,\dot{\mathbf{r}}\,, \qquad (2.10)$$

  where $b > 0$ is a constant (the *friction coefficient*), and $\dot{\mathbf{r}}$ is the particle velocity.

- A quadratic drag holds when the resistance is due to the body having to push fluid to the side as it moves, for example a rowing boat moving through water. The force is

$$\mathbf{F} = -D\,|\dot{\mathbf{r}}|\,\dot{\mathbf{r}}\,, \qquad (2.11)$$

  where the constant $D > 0$.[6]

Both are effective/approximate descriptions of the actual force on a body moving through fluid. At the molecular level the force arises due to collisions between the body and the fluid particles (with the fluid particles also colliding with each other). These molecular forces are ultimately electromagnetic forces.

**Example** (Linear drag): Consider a particle falling under gravity with a linear drag force. The particle is released from rest at time $t = 0$.

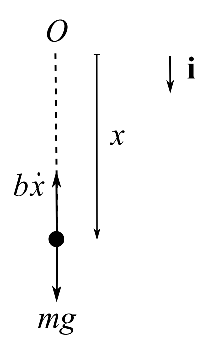



Figure 4: A particle falling under gravity with a linear drag.

We choose an inertial frame with origin $O$, and (unconventionally) take $\mathbf{i}$ to be a unit vector in the downwards direction. We measure the position of the particle from $O$ in the direction $\mathbf{i}$ by the coordinate $x$. See Figure 4. The force due to gravity is $mg\,\mathbf{i}$, while the drag force is in the opposite

[6]* The constant $D$ depends on the density of the fluid and the cross-sectional area of the body.

direction to the velocity, meaning this is $-b\,\dot{x}\,\mathbf{i}$ where the friction coefficient $b > 0$. Newton's second law reads

$$m\ddot{x}\,\mathbf{i} \;=\; mg\,\mathbf{i} - b\,\dot{x}\,\mathbf{i}\ . \tag{2.12}$$

We hence deduce the one-dimensional equation

$$m\ddot{x} \;=\; mg - b\,\dot{x} \qquad \Longrightarrow \qquad \ddot{x} \;=\; g - \frac{b}{m}\,\dot{x}\ . \tag{2.13}$$

We have reduced the problem to a simple ODE. There are various ways to proceed, but perhaps the most standard in this case is to first solve the related *homogeneous equation*

$$\ddot{x}_0 \;=\; -\frac{b}{m}\dot{x}_0\ . \tag{2.14}$$

This integrates immediately to

$$\dot{x}_0 \;=\; -\frac{b}{m}(x_0 - A)\ , \tag{2.15}$$

where we have made a convenient choice of integration constant $A$. We then solve this as

$$x_0(t) - A \;=\; B\,\mathrm{e}^{-\frac{b}{m}t}\ , \tag{2.16}$$

where $B$ is the second integration constant. One easily verifies that a *particular solution* to the original equation (2.13) is $x(t) = \frac{mg}{b}t$, and thus the general solution to (2.13) is

$$x(t) \;=\; A + B\,\mathrm{e}^{-\frac{b}{m}t} + \frac{mg}{b}t\ . \tag{2.17}$$

Finally, the initial conditions give $x(0) = 0 = \dot{x}(0)$, which allows us to determine the integration constants. The solution to the problem is

$$x(t) \;=\; \frac{m^2 g}{b^2}\left(\mathrm{e}^{-\frac{b}{m}t} - 1\right) + \frac{mg}{b}t\ . \tag{2.18}$$

We conclude this example with a few more remarks. First, notice that the dimensions of the friction coefficient $b$ are $[b] = \mathrm{M}\,\mathrm{T}^{-1}$, implying that the combination $\frac{b}{m}t$ is dimensionless, as it should be (it is the argument of an exponential function in the solution). Second, notice that the velocity of the particle is

$$\dot{x} \;=\; \frac{mg}{b}\left(1 - \mathrm{e}^{-\frac{b}{m}t}\right) \;\longrightarrow\; \frac{mg}{b} \quad \text{as } t \to \infty\ . \tag{2.19}$$

This is called the *terminal velocity*. In this limit the force of gravity is balanced by the viscous drag force: there is no net force on the particle, and it hence moves with constant velocity. Note that in this limit both sides of the equation of motion (2.12) are separately zero (which is the *particular solution* $x(t) = A + \frac{mg}{b}t$). ■

For a similar example with quadratic drag force, see Problem Sheet 1.

### 2.3 Hooke's law for springs

Consider a spring that is fixed at one end and attached to a point particle at the other. The particle experiences a force directed along the line of the spring which is proportional to the *extension* of the spring from its natural (equilibrium) length $l$. Taking the spring to lie along the $x$ axis, fixed at the origin, we have (see Figure 5)

$$\mathbf{F} \quad = \quad -k(x-l)\,\mathbf{i}\ , \tag{2.20}$$

where $x$ is the length of the spring and $k > 0$ is a constant called the *spring constant*. This is a *restoring force*; that is, the force opposes any motion away from the equilibrium position $x = l$. The force (2.20) is sometimes also referred to as the *spring tension*, and its magnitude is given by the general formula tension $= k \times$ (extension from natural length). Hooke's law is an *effective force*, resulting from intermolecular forces in the spring, which are ultimately electromagnetic forces. In fact the electromagnetic force is essentially responsible for almost all physical forces encountered in everyday experience, with the exception of gravity.

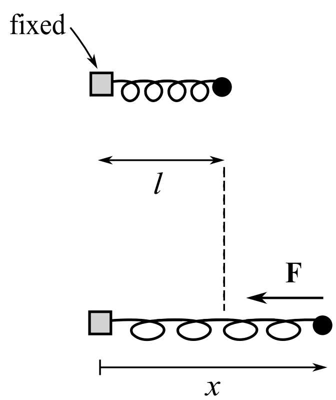



Figure 5: A spring lying along the $x$ axis, with one end fixed at the origin. The first diagram shows the equilibrium position, while the second shows the spring extended by a length $x - l > 0$, with the resulting restoring force. Notice that when $x - l < 0$, which compresses the spring rather than stretches it, the force acts in the opposite direction (to the right).

Let us consider the dynamics of the particle. All the forces acting are shown in Figure 6. We assume that the particle slides on a frictionless surface. The weight $mg$ acting downwards is balanced by an equal but opposite normal reaction $\mathbf{N}$ from the surface acting upwards.[7] The only net force acting is hence the tension in the spring, and Hooke's law (2.20) allows us to write down the equation of motion

$$m\ddot{x} \quad = \quad -k(x-l)\ . \tag{2.21}$$

[7]Warning: that these are equal and opposite forces is **not** an example of Newton's third law! The equal and opposite forces in **N3** apply to two *different* bodies, never to the same body. The other two bodies involved in this case are the Earth and the frictionless surface, respectively.

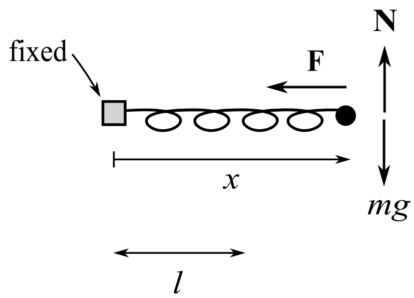



Figure 6: The forces acting on a particle attached to a spring.

A particular solution of (2.21) is $x = l$, which is the equilibrium configuration. The homogeneous equation reads

$$\ddot{x}_0 + \omega^2 x_0 \ = \ 0 \ , \tag{2.22}$$

where we have defined $\omega \equiv \sqrt{\frac{k}{m}} > 0$. The ODE (2.22) is said to describe a *simple harmonic oscillator*, with solutions being *simple harmonic motion*. The general solution to (2.22) is

$$x_0(t) \ = \ C \cos \omega t + D \sin \omega t \ = \ A \cos (\omega t + \phi) \ , \tag{2.23}$$

where both forms of the solution may be useful. Without loss of generality we may take the integration constant $A > 0$, which is called the *amplitude*, while the constant $\phi$ is called the *phase*. The motion is periodic, with period $T = 2\pi/\omega$ in $t$. The parameter $\omega$ is called the (angular) *frequency* of the oscillator. This is the simplest example of oscillatory motion. The simple harmonic oscillator is ubiquitous in mechanics, and indeed physics more generally, for the reasons explained in section 3.3.

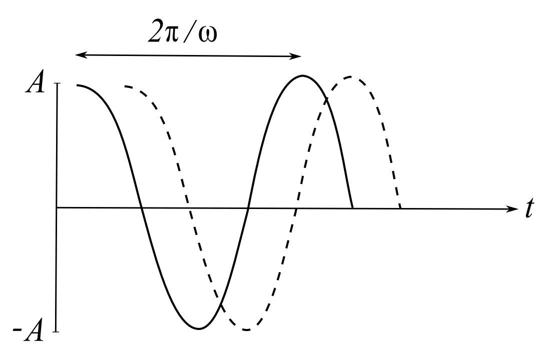



Figure 7: Simple harmonic motion with amplitude $A > 0$, and period $2\pi/\omega$. Shown are two solutions with different choices of the phase $\phi$.

Returning to the spring, the solution to (2.21) is

$$x(t) \ = \ l + C \cos \omega t + D \sin \omega t \ = \ l + A \cos (\omega t + \phi) \ . \tag{2.24}$$

The integration constants are fixed by initial conditions.

**Example**: Consider the above system at time $t = 0$ in its equilibrium position, with the particle having initial velocity $\dot{x}(0) = u$. In this case it's easier to use the first form of the solution in (2.24). Substituting $x(0) = l$ gives $C = 0$, while $\dot{x}(0) = u$ gives $D = u/\omega$, giving the solution

$$x(t) \quad = \quad l + \frac{u}{\omega} \sin \omega t \ . \tag{2.25}$$

## 2.4 Particle in an electromagnetic field

Elementary particles, in addition to having a mass, also have a property called *electric charge*. This is measured in *Coulombs* C, and the electron and proton have equal and opposite charges $q = \mp 1.60 \times 10^{-19}$ C. In general, a particle of charge $q$ moving in an *electromagnetic field* experiences a force given by the *Lorentz force law*

$$\mathbf{F} \quad = \quad q\,\mathbf{E} + q\,\dot{\mathbf{r}} \wedge \mathbf{B} \ . \tag{2.26}$$

Here $\dot{\mathbf{r}}$ is the velocity of the particle, $\mathbf{E}$ is the *electric field*, and $\mathbf{B}$ is the *magnetic field*. In general $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B} = \mathbf{B}(\mathbf{r}, t)$ depend on both position and time, making them time-dependent vector fields.

In Maxwell's theory of electromagnetism $\mathbf{E}$ and $\mathbf{B}$ become dynamical objects in their own right, satisfying their own equations of motion – *Maxwell's equations*. These equations are studied in the course B7.2. We won't need any detailed knowledge of electromagnetism for this course: the Lorentz force law (2.26) is for us simply an interesting example of a force law. Notice that in general $\mathbf{F} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$, with the dependence on the particle's velocity $\dot{\mathbf{r}}$ arising from the magnetic part of the force $\mathbf{F}_{\mathrm{mag}} = q\,\dot{\mathbf{r}} \wedge \mathbf{B}$. Due to the cross product the latter is perpendicular to both the velocity and the magnetic field, which leads to some interesting dynamics.

**Example** (Charged particle moving in a constant magnetic field): Ignoring gravity, determine the trajectory of a particle of charge $q$ moving in *constant* magnetic field $\mathbf{B}$.

The force on the particle is given by the Lorentz force law (2.26), which gives $\mathbf{F} = q\,\dot{\mathbf{r}} \wedge \mathbf{B}$. Hence Newton's second law reads

$$m\ddot{\mathbf{r}} \quad = \quad q\,\dot{\mathbf{r}} \wedge \mathbf{B} \ . \tag{2.27}$$

Since $\mathbf{B}$ is constant we may immediately integrate this with respect to time $t$:

$$m\dot{\mathbf{r}} \quad = \quad q\,\mathbf{r} \wedge \mathbf{B} + m\mathbf{V} \ . \tag{2.28}$$

The last term is the integration constant (or three of them, given that (2.27) is a vector equation). We have chosen the integration constant so that at time $t = 0$ the particle is at the origin $\mathbf{r} = \mathbf{0}$ and has velocity $\dot{\mathbf{r}} = \mathbf{V}$ – notice that all we have done here is made a convenient choice of origin. Moreover, without loss of generality we may further choose the magnetic field to point along the $z$

axis, so $\mathbf{B} = (0, 0, B)$, and then use the freedom to rotate the $(x, y)$ plane so that $\mathbf{V} = (V_1, 0, V_3)$. Writing $\mathbf{r} = (x, y, z)$, note that $\mathbf{r} \wedge \mathbf{B} = -xB\,\mathbf{j} + yB\,\mathbf{i}$. Writing the integrated equation of motion (2.28) out in components thus gives the three ODEs

$$
\begin{aligned}
m\dot{x} &= qB\,y + mV_1\ , \\
m\dot{y} &= -qB\,x\ , \\
m\dot{z} &= mV_3\ .
\end{aligned}
\tag{2.29}
$$

The last equation immediately solves to give $z(t) = V_3 t$ (using the initial condition $\mathbf{r}(0) = \mathbf{0}$). Solving for $x$ in terms of $\dot{y}$ from the second equation and substituting into the first gives a second order ODE for $y$. One can solve the equations this way, but a slicker way to proceed is to introduce the complex variable $\zeta = x + \mathrm{i}y$. Specifically, taking the first equation in (2.29) and adding i times the second equation gives the *complex* equation

$$
m(\dot{x} + \mathrm{i}\dot{y}) = -qB\,\mathrm{i}\,(x + \mathrm{i}y) + mV_1\ , \tag{2.30}
$$

which in terms of $\zeta = x + \mathrm{i}y$ reads

$$
m\dot{\zeta} = -qB\,\mathrm{i}\zeta + mV_1\ . \tag{2.31}
$$

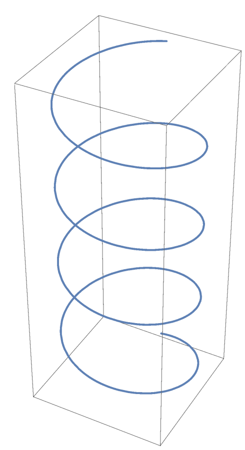


Figure 8: The path of a charged particle in a constant magnetic field (in the vertical direction).

Defining

$$
\omega = \frac{qB}{m}\ . \tag{2.32}
$$

we may rewrite (2.31) as

$$
\dot{\zeta} = -\mathrm{i}\omega\,\zeta + V_1 = -\mathrm{i}\omega\left(\zeta + \frac{\mathrm{i}}{\omega}V_1\right)\ . \tag{2.33}
$$

This is easily solved to give

$$
\zeta(t) + \frac{\mathrm{i}}{\omega}V_1 = \alpha\,\mathrm{e}^{-\mathrm{i}\omega t}\ , \tag{2.34}
$$

where $\alpha$ is a complex integration constant. Using the initial condition $\zeta(0) = 0$ fixes $\alpha = \mathrm{i}V_1/\omega$. Writing $\mathrm{e}^{-\mathrm{i}\omega t} = \cos\omega t - \mathrm{i}\sin\omega t$ the solution hence reads

$$x(t) + \mathrm{i}y(t) \quad = \quad \zeta(t) \; = \; \frac{\mathrm{i}V_1}{\omega}\,(\cos\omega t - \mathrm{i}\sin\omega t - 1) \ . \tag{2.35}$$

Recalling that $z(t) = V_3 t$, the trajectory of the particle is hence

$$\mathbf{r}(t) \quad = \quad \big(x(t), y(t), z(t)\big) \; = \; \left(\frac{V_1}{\omega}\sin\omega t\,,\ -\frac{V_1}{\omega} + \frac{V_1}{\omega}\cos\omega t\,,\ V_3 t\right) \ . \tag{2.36}$$

The frequency $\omega$ defined by (2.32) is called the *cyclotron frequency*. The trajectory traces out a helix, shown in Figure 8. Notice that the projection of this to the $(x, y)$ plane is a circle of radius $V_1/\omega$, with the time taken to complete a circle being $2\pi/\omega$. ■

# 3 Motion in one dimension

In the previous section we were always able to solve Newton's second law explicitly, in closed form. Unfortunately, as soon as we move beyond the simplest examples, for example by combining the effects of different forces, it becomes very difficult to solve for the trajectory explicitly. In this section we introduce some general methods that help to understand certain aspects of the dynamics, without having to solve Newton's second law directly. We will here focus (mainly) on dynamics in one dimension. Why focus on one-dimensional motion when the real world is three-dimensional? Firstly, the problems are simpler, and when studying any new subject one should always begin by trying to isolate the new phenomena and features in their simplest setting. But more importantly, many three-dimensional problems may effectively be reduced to studying lower dimensional problems.

## 3.1 Energy

Consider a particle moving along the $x$ axis, subject to a force $F = F(x)$ that depends only on the particle's position $x$. Newton's second law gives

$$m\ddot{x} \;=\; F(x)\;. \tag{3.1}$$

This is a second order ODE, but in this case there always exists a first integral. To see this, we first introduce:

**Definition** The *kinetic energy* of the particle is $T = \frac{1}{2}m\dot{x}^2$. We may also write this in terms of momentum $p = m\dot{x}$ as $T = p^2/2m$. Energy is measured in *Joules* J, with 1 J = 1 $\mathrm{kg\,m^2\,s^{-2}}$.

To see the utility of this, we calculate

$$\dot{T} \;=\; m\dot{x}\,\ddot{x} \;=\; F(x)\,\dot{x}\;, \tag{3.2}$$

where the second equality uses (3.1). Suppose the particle starts at position $x_1$ at time $t_1$, and finishes at $x_2$ at time $t_2$. Integrating (3.2) with respect to time $t$ gives

$$T(t_2) - T(t_1) \;=\; \int_{t_1}^{t_2} \dot{T}\,\mathrm{d}t \;=\; \int_{t_1}^{t_2} F(x(t))\,\dot{x}\,\mathrm{d}t \;=\; \int_{x_1}^{x_2} F(x)\,\mathrm{d}x\;. \tag{3.3}$$

This motivates another definition:

**Definition** The *work done* $W$ by the force in moving the particle from $x_1$ to $x_2$ is

$$W \;=\; \int_{x_1}^{x_2} F(x)\,\mathrm{d}x\;. \tag{3.4}$$

Equation (3.3) thus proves:

**Work-Energy Theorem** The work done by the force is the change in kinetic energy:

$$W = T(t_2) - T(t_1) \ . \tag{3.5}$$

■

This notion of *work* also leads to the following definition:

**Definition** The *potential energy* of the particle is $V(x) = -\int_{x_0}^{x} F(y)\,\mathrm{d}y$, where $x_0$ is arbitrary.

By definition, the potential energy $V(x)$ is *minus* the work done by the force in moving the particle from $x_0$ to $x$. This *a priori* depends on the choice of $x_0$, but if we change $x_0 \mapsto \tilde{x}_0$ the potential energy changes to $V(x) \mapsto V(x) - \int_{\tilde{x}_0}^{x_0} F(y)\,\mathrm{d}y$. Changing $x_0$ thus simply shifts $V(x)$ by an additive constant: *potential energy is understood to be defined only up to an overall additive constant.*

Using the Fundamental Theorem of Calculus we may write the force as

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x} = -V'(x) \ . \tag{3.6}$$

**Examples**:

1. For $F = -mg$ a choice of potential is $V(x) = mgx$.
2. For Hooke's linear force $F = -k(x-l)$ a choice of potential is $V(x) = \frac{1}{2}k(x-l)^2$.

Notice that we've made a natural choice of additive integration constant in each case, but any choice will do. Also, be careful with the signs!

**Conservation of Energy Theorem** The *total energy* of the particle

$$E = T + V \tag{3.7}$$

is *conserved*, *i.e.* is constant when evaluated on a solution to Newton's second law (3.1).

**Proof 1**: From the Work-Energy Theorem we already have

$$T(t_2) - T(t_1) = W = \int_{x_1}^{x_2} F(x)\,\mathrm{d}x = V(x_1) - V(x_2) \ . \tag{3.8}$$

Rearranging thus gives

$$E = T(t_1) + V(x_1) = T(t_2) + V(x_2) \ . \tag{3.9}$$

Since the initial and final positions and times here are arbitrary, this proves $E$ is conserved. ■

**Proof 2**: More precisely we first write the right hand side of (3.7) as $T(t) + V(x(t))$. Using the chain rule we then have

$$\dot{E} \;=\; \dot{T} + \dot{V} \;=\; m\dot{x}\,\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \;=\; \dot{x}\,(m\ddot{x} - F)\;. \tag{3.10}$$

It follows that $\dot{E} = 0$ is implied by Newton's second law.[8] ■

The fact that $E$ is constant implies that in the motion any loss of potential energy necessarily results in an equal gain in the kinetic energy $T = \frac{1}{2}m\dot{x}^2$, and hence a gain in the speed $|\dot{x}|$ of the particle (and of course the same statement with loss/gain interchanged).

Notice that we may rewrite (3.7) as

$$\frac{1}{2}m\dot{x}^2 \;=\; E - V(x)\;. \tag{3.11}$$

This equation has many implications. First, knowing the energy $E$ and position of the particle immediately gives its speed $|\dot{x}|$. Second, since kinetic energy $T = \frac{1}{2}m\dot{x}^2 \geq 0$ is non-negative, we always have $V(x) \leq E$. This confines the possible location of the particle, for fixed energy. We'll see in section 3.2 that this allows us to determine the *qualitative* motion of particles, in a general potential $V(x)$. But we may also obtain quantitative information.

**Example** (Maximum height under gravity (again)): Let's revisit the example in section 1.3: consider a particle moving vertically under gravity, which at time $t = 0$ starts at height $z = 0$ with velocity $\dot{z} = u > 0$ upwards. What is the maximum height of the particle?

The potential is $V(z) = mgz$. The conserved energy $E$ may be calculated from the initial conditions, which gives $E = T(0) = \frac{1}{2}mu^2$. Thus (3.11) reads

$$\frac{1}{2}m\dot{z}^2 \;=\; \frac{1}{2}mu^2 - mgz\;. \tag{3.12}$$

The maximum height occurs when $\dot{z} = 0$, which immediately gives

$$z_{\mathrm{max}} \;=\; \frac{u^2}{2g}\;. \tag{3.13}$$

■

We may also write that the work done in moving the particle from position $x_0$ at time $t_0$ to position $x$ at time $t$ is

$$W(t) \;=\; \int_{x_0}^{x(t)} F(y)\,\mathrm{d}y \;=\; V(x_0) - V(x(t))\;. \tag{3.14}$$

[8]* Notice that conversely $\dot{E} = 0$ implies Newton's second law, unless $\dot{x}$ is zero for all time. In the latter case $x = x_0$ is constant, and equation (3.11) then implies that $E = V(x_0)$. This solves (3.11), but Newton's second law only holds if in addition $F(x_0) = -V'(x_0) = 0$.

**Definition** *Power* $P$ = rate of work done, so that

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = F\,\dot{x} \,. \tag{3.15}$$

Power is measured in *Watts*, with 1 Watt = 1 J s$^{-1}$.

For *conservative* forces, meaning there is a potential satisfying (3.6), the work done by the force in any motion can be positive or negative, in the former case causing a corresponding increase in kinetic energy, by the Work-Energy Theorem. However, this is in general not the case for time-dependent or velocity-dependent forces. For example, for a linear drag force $F = -b\,\dot{x}$, with $b > 0$, the work done by the resistive force over a small distance $\delta x$ is $-b\,\dot{x}\,\delta x = -b\,\dot{x}^2\,\delta t < 0$. Thus the work done is *always negative*, no matter what the motion. For a *dissipative force*, such as drag or friction, energy is apparently lost. However, at a microscopic level energy should be conserved – this is believed to be a fundamental principle in physics. In the case of fluid drag, the issue is that we have ignored the "back-reaction" of our body on the fluid particles. In each collision between the body and the fluid particles energy is conserved, but some of the kinetic energy is transferred to the fluid particles, increasing their average kinetic energy. But by definition this means we lost kinetic energy of our object as heat – the fluid will be a bit warmer.

## 3.2 Motion in a general potential

Returning to equation (3.11), slightly rerranging gives us

$$\dot{x}^2 = \frac{2}{m}(E - V(x)) \,. \tag{3.16}$$

This is a *first order ODE*, which we can in principle solve as

$$t = \pm \int \frac{\mathrm{d}x}{\sqrt{\frac{2}{m}(E - V(x))}} \,. \tag{3.17}$$

This gives $t$ as a function of $x$. Assuming we can do the integral on the right hand side, we can invert the relation to find $x(t)$. The problem here is that, apart from in very simple problems, we usually can't evaluate the integral. Of course, what this means is that we can't write it in terms of known elementary functions; but some of these integrals are so important, they are used as the *definition* of new functions.

**Example** (Quartic potential): Consider a general *quartic* potential $V(x) = -\sum_{k=1}^{4} \frac{1}{k} a_{k-1}\, x^k$, where the $a_k$ are constant. Newton's second law reads

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \,, \tag{3.18}$$

with an arbitrary cubic force on the right hand side. The integral on the right hand side of (3.17) is called an *elliptic integral*. Using (3.17) we must then invert this to find $x(t)$ that solves the equation of motion (3.18). The inverse of an elliptic integral is called an *elliptic function*. These appear repeatedly in mathematics, and are in themselves a beautiful topic, with surprising features. ■

**Example** (Quadratic potential – the harmonic oscillator): A special case of the former example is a *quadratic* potential, with $a_2 = a_3 = 0$. We have already solved this problem in section 2.3: it is just the spring – see equation (2.21). Let us begin with the homogeneous harmonic oscillator equation (2.22)

$$\ddot{x} + \omega^2 x \quad = \quad 0 \ . \tag{3.19}$$

The force acting is $F(x) = -m\omega^2 x$, which has a potential energy function $V(x) = \frac{1}{2}m\omega^2 x^2$. Equation (3.17) hence reads

$$t \quad = \quad \pm \int \frac{\mathrm{d}x}{\omega\sqrt{\frac{2E}{m\omega^2} - x^2}} \ . \tag{3.20}$$

We may solve this by making the substitution

$$x \quad = \quad \sqrt{\frac{2E}{m\omega^2}} \cos\theta \ , \tag{3.21}$$

which gives

$$t \quad = \quad \mp \int \frac{1}{\omega}\mathrm{d}\theta \qquad \Longrightarrow \qquad t - t_0 \quad = \quad \mp\frac{1}{\omega}\cos^{-1}\left(\frac{x}{\sqrt{2E/m\omega^2}}\right) \ . \tag{3.22}$$

Here $t_0$ is an integration constant. The solution is hence simple harmonic motion

$$x(t) \quad = \quad \sqrt{\frac{2E}{m\omega^2}} \cos\left[\omega(t - t_0)\right] \ . \tag{3.23}$$

Notice that in this case it is *easier* to solve the second order equation of motion, than to integrate the first order conservation of energy equation! On the other hand, we have learned that the amplitude $A = \sqrt{2E/m\omega^2}$, *c.f.* equation (2.23). ■

Let's now consider a particle moving in a *general* potential $V(x)$. An illustrative example is shown in Figure 9. In general we won't be able to do the integral in (3.17), nor will we be able to explicitly solve Newton's second law. However, we can deduce quite a lot about the *qualitative* motion, using only the fact that $E = T + V$ is conserved, and $T \geq 0$, for different values of the conserved energy $E$.

- Referring to Figure 9, suppose our particle has energy $E_0$, and starts its motion at some $x > x_0$ with $\dot{x} < 0$. Since $\dot{x}$ is negative the particle will start out moving to the left, but as it does so $T = E_0 - V$ *decreases* to zero as it approaches $x_0$, where by definition $V(x_0) = E_0$. At $x_0$ the particle has zero kinetic energy $T = 0$, and so is momentarily at rest. However, since $F(x_0) = -V'(x_0) > 0$ at this point there is a force acting to the right. The particle's motion hence turns around at $x_0$ to have $\dot{x} > 0$ for $x > x_0$. Since $T = E_0 - V > 0$ for $x > x_0$, the particle continues to move to the right (and in fact escapes to $x \to \infty$).

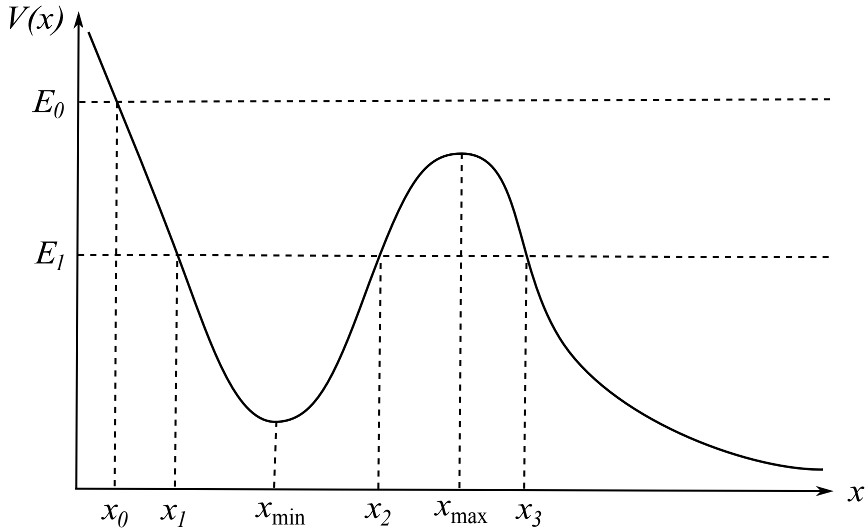



Figure 9: A general potential $V(x)$, with various points marked on the $x$ axis. $x_{\min}$ and $x_{\max}$ are a local minimum and local maximum, respectively. At any point $x$ the force acting on the particle is *minus the slope of the potential*, $F(x) = -V'(x)$.

- For $E = E_1$ and $x > x_3$ the discussion is similar to that above. However, if the particle begins its motion at $x \in [x_1, x_2]$, it must remain bounded in this interval for all time – we say it has insufficient energy to escape the "potential well". At $x = x_1$ or $x = x_2$ note that again $T = 0$ and the particle is momentarily at rest. However, $F(x_1) > 0$ while $F(x_2) < 0$, meaning that the particle simply bounces back and forth inside the interval $[x_1, x_2]$.

For $E = E_1$ the regions $x < x_1$ and $x_2 < x < x_3$ are *classically forbidden* – the particle doesn't have enough energy to exist at these points. Notice that at $x = x_{\min}$ or $x = x_{\max}$ we have $F(x) = -V'(x) = 0$ and the particle momentarily has no force acting on it (more on this in the next subsection).

* In *quantum mechanics* there is a non-zero (but exponentially small) probability that a particle in the potential well $x \in [x_1, x_2]$ can "quantum tunnel" through the hill between $x_2$ and $x_3$, and escape to $x \to \infty$. You can study such strange quantum behaviour in Part A Quantum Theory.

## 3.3 Motion near equilibrium

Given a dynamical system, one of the first questions we might ask is: are there any *equilibrium configurations*? By definition, if you put the system in such a configuration, it will stay there. Here is a more formal definition, in our setting of one-dimensional motion on the $x$ axis:

**Definition** An *equilibrium configuration* is a solution to Newton's second law (3.1) with $x = x_e =$ constant. Since this implies $\ddot{x} = 0$ for all time $t$, Newton's second law implies that $F(x_e) = 0$, and there is no net force acting on the particle.

For a conservative force $0 = F(x_e) = -V'(x_e)$ implies that $x_e$ is a *critical point* of the potential $V(x)$.

Consider motion near an equilibrium point $x = x_e$. We may begin by expanding Newton's second law around this point (assuming $F(x)$ is suitably analytic):

$$m\ddot{x} \;=\; F(x) \;=\; F(x_e) + (x - x_e)F'(x_e) + O((x - x_e)^2) \; . \tag{3.24}$$

By definition we have $F(x_e) = 0$. We change variables to $\xi \equiv x - x_e$, so that the equilibrium point is now at $\xi = 0$. Assuming we are sufficiently close to the latter, so that the quadratic and higher order terms in (3.24) are small, we may write down the following *approximate linear differential equation* for $\xi$:

$$m\ddot{\xi} \;=\; F'(x_e)\xi \; . \tag{3.25}$$

**Definition** Equation (3.25) is called the *linearized equation of motion.* Solutions to this linear homogeneous equation are called *linearized solutions.*

There are three qualitatively different cases, depending on the sign of the constant

$$K \;\equiv\; -F'(x_e) \; . \tag{3.26}$$

- $K > 0$

  In this case we may define $\omega = \sqrt{K/m} > 0$. The linearized equation of motion (3.25) then reads $\ddot{\xi} + \omega^2\xi = 0$, which is the simple harmonic oscillator we solved in section 2.3. The general solution is $\xi(t) = A\cos(\omega t + \phi)$. In this case $\xi = 0$ is called a *point of stable equilibrium* – for amplitude $A$ small enough so that it is consistent to ignore the higher order terms in the expansion of the force (3.24), the system executes small oscillations around the equilibrium point. The frequency of these oscillations is $\omega$. Crucially, this analysis applies to *any* point of stable equilibrium, and it is for this reason that the harmonic oscillator is so important.

  **Example** (Hooke's law): We now see why Hooke's law for springs isn't really a fundamental law of physics at all – it follows simply from the fact that the system is near a stable equilibrium. ■

- $K < 0$

  In this case we may define $p = \sqrt{-K/m} > 0$. The linearized equation of motion (3.25) now reads

  $$\ddot{\xi} - p^2\xi \;=\; 0 \; , \tag{3.27}$$

  which has general solution

  $$\xi(t) \;=\; A\,\mathrm{e}^{pt} + B\,\mathrm{e}^{-pt} \; , \tag{3.28}$$

with $A$ and $B$ integration constants. A generic small displacement of the system at time $t = 0$ will have both $A$ and $B$ non-zero, and the solution grows exponentially with $t$, for both $t > 0$ and $t < 0$. The higher order terms in the Taylor expansion, that we ignored, quickly become relevant. Such equilibria are hence termed *unstable*.

- $K = 0$

  Finally, if $K = 0$ the first two terms in the Taylor expansion in (3.24) are zero, and we need to expand to higher order to determine what happens (although not in this course!).

We may rephrase all of the above discussion in terms of potentials. We similarly expand

$$V(x) \quad = \quad V(x_e) + (x - x_e)V'(x_e) + \frac{1}{2}(x - x_e)^2 V''(x_e) + O((x - x_e)^3) \; . \tag{3.29}$$

Without loss of generality we may choose the arbitrary additive constant in $V$ so that $V(x_e) = 0$. Moreover, $V'(x_e) = -F(x_e) = 0$. This means that near equilibrium the potential is approximately quadratic:

$$V_{\text{quad}}(x) \quad = \quad \frac{1}{2}K(x - x_e)^2 \; , \tag{3.30}$$

where $K = V''(x_e) = -F'(x_e)$, as in (3.26). A stable equilibruim point with $K > 0$ is then a *local minimum* of the potential (for example $x_e = x_{\text{min}}$ in Figure 9). An unstable equilibruim point with $K < 0$ is instead a *local maximum* (for example $x_e = x_{\text{max}}$ in Figure 9).

Let's see how to use some of these ideas in a realistic example (*i.e.* an exam question!):

**Example** (Taken from the Mods Examination paper, 2003): A bead of mass $m$ slides along a smooth, straight horizontal wire which passes through the origin $O$. The bead is attached to a light, straight elastic spring of natural length $l$ and spring constant $k$, and the other end of the spring is attached to a fixed point $P$ which is a distance $d$ vertically above $O$.

(i) If $x$ denotes the coordinate of the bead, relative to $O$, explain why the tension in the spring is $\mathbb{T} = k\left(\sqrt{d^2 + x^2} - l\right)$, and show that

$$\ddot{x} \quad = \quad \frac{k}{m}x\left(\frac{l}{\sqrt{d^2 + x^2}} - 1\right) \; . \tag{3.31}$$

(ii) Find the equilibrium solutions of this equation, and determine whether they are stable or unstable, distinguishing carefully between the two cases $l < d$ and $l > d$.

**Solution**: The set up is shown in Figure 10. From Pythagoras' Theorem the extension of the spring from its natural length is $\sqrt{d^2 + x^2} - l$, and from Hooke's law the tension $\mathbb{T}$ is the spring

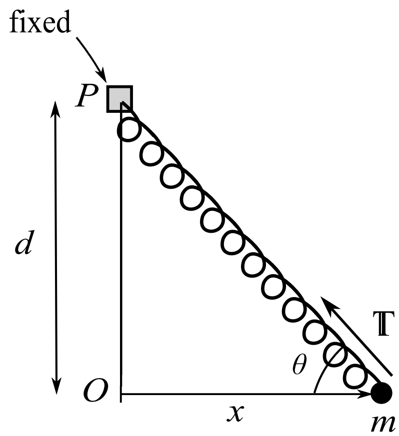



Figure 10: The spring-bead system. The bead of mass $m$ is constrained to move along the $x$ axis.

constant $k$ times this extension.[9] Writing down the component of Newton's second law in the $x$ direction gives

$$m\ddot{x} \quad = \quad -\mathbb{T}\cos\theta \quad = \quad -\frac{\mathbb{T}x}{\sqrt{d^2+x^2}} \quad \equiv \quad F(x) \ . \tag{3.32}$$

Substituting the given expression for $\mathbb{T}$ and rearranging slightly then gives the equation of motion (3.31) for $x$.

Equilibrium solutions have the right hand side of (3.32) equal to zero, namely $F(x_e) = 0$ where

$$F(x) \quad = \quad kx\left(\frac{l}{\sqrt{d^2+x^2}} - 1\right) \ . \tag{3.33}$$

The zeros are at $x_e = 0$ and $x_e = x_0$, where $l = \sqrt{d^2 + x_0^2}$. Notice this solves as

$$x_0 \quad = \quad \pm\sqrt{l^2 - d^2} \ , \tag{3.34}$$

and this makes sense only if $l \geq d$. Note also that the set up is symmetric under taking $x \mapsto -x$, so the behaviour of both equilibria in (3.34) should be the same. One computes

$$F'(x_e) \quad = \quad k\left(\frac{l}{\sqrt{d^2+x_e^2}} - 1\right) - \frac{x_e^2\, kl}{(d^2+x_e^2)^{3/2}} \ . \tag{3.35}$$

In particular

$$F'(0) \quad = \quad k\left(\frac{l}{d} - 1\right) \ , \tag{3.36}$$

so that the equilibrium at $x_e = 0$ is stable if $l < d$ and unstable if $l > d$. On the other hand

$$F'(x_0) \quad = \quad -\frac{x_0^2\, kl}{\left(d^2+x_0^2\right)^{3/2}} < 0 \ , \tag{3.37}$$

[9] It is unfortunate that the letter $\mathbb{T}$ is variously used to denote kinetic energy, tension, and periods of oscillation. However, these all have different dimensions, and the context should always be clear.

implying that $x_0$ only exists as a distinct equilibrium when $l > d$, and in this case it is stable. ■

**Remark**: You might ask: what about the component of Newton's second law in the vertical direction? In particular, what balances the vertical force $\mathbb{T}\sin\theta$ to constrain the bead to move only along the $x$ axis? This is an example of a *constraint force*, studied in detail in section 5. Revisit this example after we cover that section, and ask yourself these questions again!

## 3.4 * Damped motion

This subsection is starred: the material is not explicitly on the syllabus, and we are unlikely to have time to cover it in lectures. However, the discussion naturally follows on from that in the previous subsection, the equations of motion may be solved explicitly, and the dynamics is interesting.

We've seen that any system near stable equilibrium is described by simple harmonic motion. More realistically, in practical applications there will be energy loss; or, as we've already commented, more accurately mechanical energy will be converted to other forms of energy (typically heat), that is not apparent in our description of the system. To model this we must assume that the force $F = F(x, \dot{x})$ depends on both position $x$ and velocity $\dot{x}$. For small displacements we may treat both of these as small, neglecting the quadratic terms $x^2$, $x\dot{x}$, $\dot{x}^2$, and higher order terms in a Taylor expansion. This leads to the *damped harmonic oscillator*, with force

$$F = -kx - b\,\dot{x} . \tag{3.38}$$

We assume that $b > 0$ and $k > 0$, so that the term $-b\,\dot{x}$ damps the motion (see the discussion after (3.15)) of a stable equilibrium point (hence $k > 0$) at $x = 0$. Newton's second law is

$$m\ddot{x} + b\,\dot{x} + kx = 0 . \tag{3.39}$$

We seek solutions of the form $x(t) = \mathrm{e}^{pt}$. Substituting this into (3.39) gives the quadratic equation

$$mp^2 + bp + k = 0 , \tag{3.40}$$

which has roots

$$p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2} . \tag{3.41}$$

Here we have defined the new parameters

$$\gamma = \frac{b}{2m} , \qquad \omega_0 = \sqrt{\frac{k}{m}} = \text{frequency of undamped oscillator} . \tag{3.42}$$

There are three cases:

#### Large damping

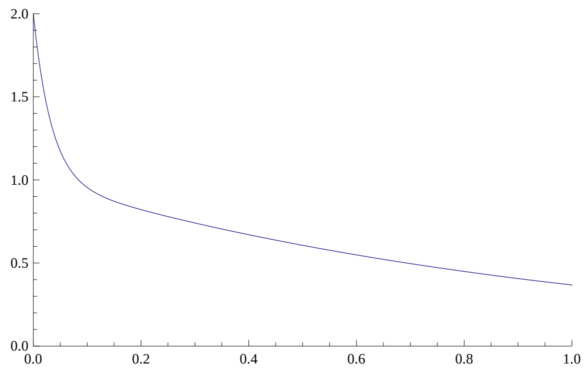
For large $b$, so that $\gamma > \omega_0$, both roots in (3.41) are real and negative:

$$p = -\gamma_{\pm} , \qquad \text{where} \qquad \gamma_{\pm} = \gamma \pm \sqrt{\gamma^2 - \omega_0^2} > 0 . \tag{3.43}$$

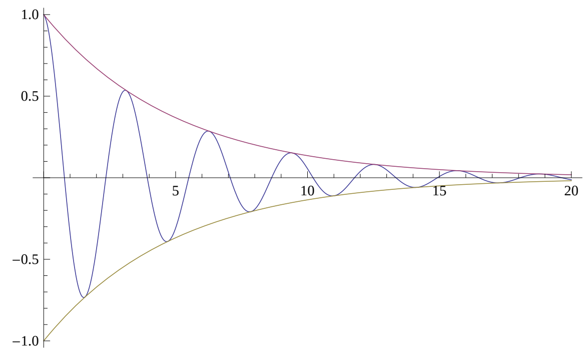
The general solution is hence

$$x(t) \quad = \quad A\,\mathrm{e}^{-\gamma_+ t} + B\,\mathrm{e}^{-\gamma_- t} \; . \tag{3.44}$$

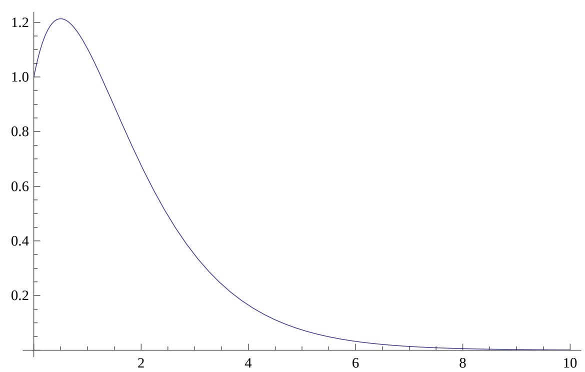
An initial displacement of the system thus tends to zero exponentially quickly. In fact the damping is so strong in this case that the oscillatory nature of the undamped oscillator has been completely swamped. Notice that provided $B \neq 0$ (true for generic initial conditions) it is the second term in (3.44), with $\gamma_- < \gamma_+$ in the exponent, that dominates – see Figure 11a.




(a) Large damping. Notice the sharp bend at around $t = 0.1$, where the first term in (3.44) has rapidly decreased to almost zero. (In the plot $A = B = 1$, $\gamma_+ = 30$, $\gamma_- = 1$.)




(b) Small damping. The solution (3.46) is shown, together with the envelopes $\pm A\,\mathrm{e}^{-\gamma t}$. (In the plot $A = 1$, $\gamma = 1/5$, $\omega = 2$, $\phi = 0$.)




(c) Critical damping. (In the plot $A = 1$, $B = 2$, $\gamma = 1$.)

Figure 11: Various behaviours of a damped harmonic oscillator.

### Small damping

For small $b$, so that $\gamma < \omega_0$, the roots in (3.41) are complex conjugates of each other, and we may write

$$p \quad = \quad -\gamma \pm \mathrm{i}\,\omega \; , \qquad \text{where} \qquad \omega \;=\; \sqrt{\omega_0^2 - \gamma^2} \; . \tag{3.45}$$

This gives

$$
\begin{aligned}
x(t) &= \frac{1}{2}\alpha\, \mathrm{e}^{-\gamma t+\mathrm{i}\omega t} + \frac{1}{2}\beta\, \mathrm{e}^{-\gamma t-\mathrm{i}\omega t} = \mathrm{Re}\left[\alpha\, \mathrm{e}^{-\gamma t+\mathrm{i}\omega t}\right] \\
&= A\, \mathrm{e}^{-\gamma t} \cos(\omega t + \phi) \ , \qquad (3.46)
\end{aligned}
$$

where the relation between the integration constants in the different forms of the solution are $\alpha = A\,\mathrm{e}^{\mathrm{i}\phi}$, $\beta = A\,\mathrm{e}^{-\mathrm{i}\phi}$. The solution hence oscillates with angular frequency $\omega < \omega_0$, but with exponentially decreasing amplitude $A\mathrm{e}^{-\gamma t}$ – see Figure 11b.

Notice that there are two characteristic timescales for the damped oscillator:

- The period of the undamped oscillator, $T_0 = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{m}{k}}$ .
- The *decay time* $T_D = \dfrac{1}{\gamma} = \dfrac{2m}{b}$, which by definition is the time it takes for the amplitude to decay from its initial value by a factor of $1/\mathrm{e}$.

We may hence form a *dimensionless parameter*

$$
Q = \frac{2\pi T_D}{T_0} = \frac{\omega_0}{\gamma} = 2\sqrt{\frac{km}{b^2}} \ . \qquad (3.47)
$$

This is called the *quality factor* of the damped oscillator. $Q > 1$ and $Q < 1$ are small and large damping, respectively, while $Q = 1$ is called *critical damping*.

#### Critical damping

When $\gamma = \omega_0$ the two roots of $p$ in (3.41) coincide, giving only one solution to the original ODE (3.39). The "missing" solution is easily checked to be $x = t\,\mathrm{e}^{-\gamma t}$, giving the general solution

$$
x(t) = (A + B\,t)\,\mathrm{e}^{-\gamma t} \ . \qquad (3.48)
$$

As for large damping, there are no oscillations – see Figure 11c. Many systems are engineered to be critically damped, *e.g.* the suspension in a car. To see why, suppose that all the parameters in the damped oscillator are fixed, apart from the friction coefficient $b$ (or equivalently $\gamma$ given by (3.42)), that we are free to adjust. Then by tuning $\gamma = \omega_0$, we ensure that a generic small displacement of the corresponding critically damped system decays more rapidly than for the same system with large damping (since the exponent $\gamma = \omega_0 > \gamma_-$, where recall that the $\gamma_-$ mode in (3.44) dominates). In addition, the system just fails to oscillate. Thus if we want to damp out general oscillations of a system as quickly as possible, we should tune it to be critically damped.

### 3.5 Coupled oscillations

So far in this section we have only considered systems with one degree of freedom, *i.e.* where the motion is described by a single function $x(t)$. In this section we briefly consider the stability of systems with two degrees of freedom. The general case is described, using more powerful methods, in the course B7.1 Classical Mechanics.

Suppose we have a dynamical system described by the *coupled* ODEs

$$\ddot{x} \;=\; F(x,y)\ , \qquad \ddot{y} \;=\; G(x,y)\ , \tag{3.49}$$

where we shall assume that $F$ and $G$ are suitably analytic.[10] As in section 3.3, an *equilibrium configuration* is a solution to (3.49) with $x = x_e$, $y = y_e$ both constant. Thus $F(x_e, y_e) = 0 = G(x_e, y_e)$. To determine the stability of such an equilibrium point, we again linearize the equations of motion. This means that we write

$$x \;=\; x_e + \xi\ , \qquad y \;=\; y_e + \eta\ , \tag{3.50}$$

where $\xi$ and $\eta$ are small, and then Taylor expand the right hand sides of (3.49), leading to

$$\begin{aligned} \ddot{\xi} &= F(x_e+\xi, y_e+\eta) \;=\; F(x_e,y_e) + \xi\,\frac{\partial F}{\partial x}(x_e,y_e) + \eta\,\frac{\partial F}{\partial y}(x_e,y_e) + \cdots\ , \\ \ddot{\eta} &= G(x_e+\xi, y_e+\eta) \;=\; G(x_e,y_e) + \xi\,\frac{\partial G}{\partial x}(x_e,y_e) + \eta\,\frac{\partial G}{\partial y}(x_e,y_e) + \cdots\ , \end{aligned} \tag{3.51}$$

where $\cdots$ denote terms of quadratic and higher order in $\xi$, $\eta$. The *linearized equations of motion* are hence

$$\begin{aligned} \ddot{\xi} &= a\,\xi + b\,\eta\ , \\ \ddot{\eta} &= c\,\xi + d\,\eta\ , \end{aligned} \tag{3.52}$$

where we have introduced the constants

$$\begin{aligned} a &= \frac{\partial F}{\partial x}(x_e,y_e)\ , \qquad & b &= \frac{\partial F}{\partial y}(x_e,y_e)\ , \\ c &= \frac{\partial G}{\partial x}(x_e,y_e)\ , \qquad & d &= \frac{\partial G}{\partial y}(x_e,y_e)\ . \end{aligned} \tag{3.53}$$

One could potentially try to solve (3.52) by *e.g.* solving the first equation for $\eta$ in terms of $\xi$ (assuming $b \neq 0$), and substituting into the second equation: this gives a fourth order ODE in $\xi$. However, it is better to write (3.52) as a *matrix equation*

$$\begin{pmatrix} \ddot{\xi} \\ \ddot{\eta} \end{pmatrix} \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}\ . \tag{3.54}$$

We then seek solutions to (3.54) of the form

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} \;=\; \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mathrm{e}^{\lambda t}\ , \tag{3.55}$$

[10] Here $x$ and $y$ denote general variables, rather than Cartesian coordinates.

where $\alpha$, $\beta$ and $\lambda$ are constant. Substituting (3.55) into (3.54) and cancelling the overall factor of $e^{\lambda t}$ gives

$$\lambda^2 \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} . \tag{3.56}$$

This says that $\lambda^2$ is an *eigenvalue* of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with corresponding *eigenvector* $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. The *characteristic equation* is

$$\det \left[ \lambda^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = \lambda^4 - (a+d)\lambda^2 + (ad - bc) = 0 , \tag{3.57}$$

which gives the eigenvalues

$$\lambda^2 = \frac{1}{2}\left( a + d \pm \sqrt{(a+d)^2 - 4(ad-bc)} \right) \equiv \lambda_\pm^2 . \tag{3.58}$$

For a general system (3.49) the solutions for $\lambda^2$ in (3.58) can be complex, in general also leading to complex $\lambda$. Recall that the linearized solution (3.55) is proportional to $e^{\lambda t} = e^{\mathrm{Re}(\lambda)t} \cdot e^{\mathrm{i}\,\mathrm{Im}(\lambda)t}$. The imaginary part of $\lambda$ determines the oscillatory part of the solution, while the real part of $\lambda$ determines the time-dependent amplitude.[11] Notice that we may take either sign for the square root in solving (3.58) for $\lambda$, implying in general 4 solutions $\pm\lambda_\pm$. This is the number we expect for two coupled second order ODEs (3.49). Let's look at a simple example.

**Example**: Consider two particles, each of mass $m$, attached to three springs, as shown in Figure 12. The springs have equilibrium length $l$ and spring constants $k$, and lie on a line. One end of the first spring is fixed, while the other end is attached to a particle of mass $m$. This mass is in turn attached to one end of the second spring, with the other end attached to a second particle of mass $m$. Finally, this second mass is attached to one end of the third spring, with the other end fixed. We denote the horizontal displacement of the first mass from its equilibrium position by $x$, and similarly the horizontal displacement of the second mass by $y$.

By Hooke's law the forces shown in Figure 12 are (careful with signs!)

$$F_1 = kx , \qquad F_2 = k(y-x) , \qquad F_3 = -ky . \tag{3.59}$$

Applying Newton's second law for each particle thus gives

$$\begin{aligned} m\ddot{x} &= F_2 - F_1 = k(y - 2x) , \\ m\ddot{y} &= F_3 - F_2 = k(x - 2y) . \end{aligned} \tag{3.60}$$

Comparing to the general formulae (3.49), (3.52) we see that the equations are already linear, and that there is one equilibrium point at $x = y = 0$. Thus in this case we may identify $x = \xi$, $y = \eta$.

[11]* If you read section 3.4, compare/contrast this with the discussion of the damped oscillator around equation (3.40).

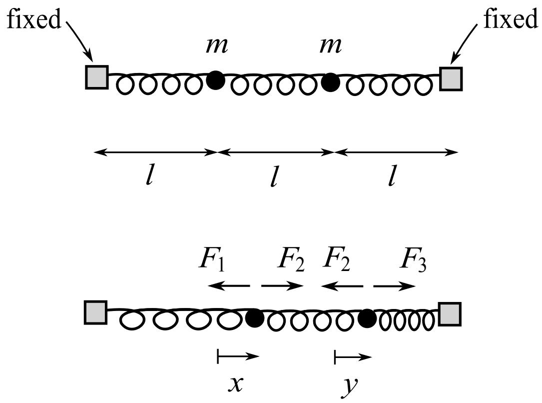



Figure 12: The system of masses and springs. The upper diagram shows the equilibrium configuration. In the lower diagram we have shown the horizontal displacements $x$ and $y$ of the two masses from their equilibrium positions, together with the various Hooke's law forces $F_1$, $F_2$, $F_3$ acting.

In matrix form (3.60) reads

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} -\frac{2k}{m} & \frac{k}{m} \\ \frac{k}{m} & -\frac{2k}{m} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} . \tag{3.61}$$

Comparing to (3.54) we read off

$$a = d = -\frac{2k}{m} , \qquad b = c = \frac{k}{m} , \tag{3.62}$$

and hence from (3.58) that

$$\lambda^2 = \frac{k}{2m}(-4 \pm 2) \qquad \Longrightarrow \qquad \lambda = \pm \mathrm{i}\sqrt{\frac{k}{m}} , \qquad \pm \mathrm{i}\sqrt{\frac{3k}{m}} . \tag{3.63}$$

Since the linearized modes (3.55) are proportional to $\mathrm{e}^{\lambda t}$, and in this case all $\lambda$ are purely imaginary, the corresponding solutions are hence oscillatory. The two values of $\lambda^2$ in (3.63) correspond to the two eigenvectors $(1, \pm 1)^{\mathrm{T}}$ of the matrix in (3.61), respectively. ■

Returning to the general case, this motivates the following definition:

**Definition** If all solutions for $\lambda = \pm\lambda_\pm$ given by (3.58) are *purely imaginary* (equivalently both $\lambda_\pm^2 < 0$), we say the equilibrium point is *stable*. We write $\lambda = \pm \mathrm{i}\omega_\pm$, where $\omega_\pm > 0$ are called the *normal frequencies* of the system. Writing $\mathrm{e}^{\lambda t} = \mathrm{e}^{\pm \mathrm{i}\omega_\pm t}$ in terms of trigonometric functions, the linearized solution is

$$\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \begin{pmatrix} \alpha_+ \\ \beta_+ \end{pmatrix} \cos(\omega_+ t + \phi_+) + \begin{pmatrix} \alpha_- \\ \beta_- \end{pmatrix} \cos(\omega_- t + \phi_-) , \tag{3.64}$$

where $\begin{pmatrix} \alpha_\pm \\ \beta_\pm \end{pmatrix}$ are the eigenvectors corresponding to the eigenvalues $\lambda_\pm^2$, respectively, and $\phi_\pm$ are constants. The solution for a given eigenvector is called a *normal mode*.

**Example**: For the system of masses and springs, the normal frequencies are $\omega_+ = \sqrt{k/m}$, $\omega_- = \sqrt{3k/m}$. The general solution is hence

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} A \cos\left(\sqrt{\frac{k}{m}} t + \phi\right) + \begin{pmatrix} 1 \\ -1 \end{pmatrix} B \cos\left(\sqrt{\frac{3k}{m}} t + \theta\right) , \qquad (3.65)$$

where $A$, $B$, $\phi$ and $\theta$ are constants. The lower frequency $\omega_+$ normal mode has the two masses oscillating together, while the higher frequency $\omega_-$ normal mode has the two masses oscillating in opposite directions. ■

The essential point of (3.64) is that near a stable equilibrium point the system behaves like two *independent* one-dimensional harmonic oscillators, of frequencies $\omega_\pm$. By solving for the eigenvalues and eigenvectors of the matrix in (3.54) we have essentially diagonalized the motion, with each normal mode being simple harmonic motion. A general perturbation (3.64) of the system is a linear combination of these two modes.

Finally, notice that if any eigenvalue $\lambda$ has a non-zero real part there will be an exponentially growing mode, with amplitude proportional to $\mathrm{e}^{\mathrm{Re}(\lambda)\,t}$ with $\mathrm{Re}(\lambda) > 0$, and the equilibrium point will be unstable.

# 4 Motion in higher dimensions

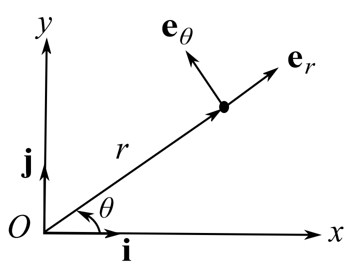
In this section we develop some general formalism that is useful for analysing dynamics in two and three dimensions. In particular in sections 4.2 and 4.3 we introduce *conservative forces* and *central forces*, respectively. The dynamics for each of these forces leads to a *conserved quantity*, *i.e.* a quantity that is constant during the motion. Conserved quantities are very important in dynamics: by definition one has at least partially integrated the equations of motion whenever one finds a conserved quantity. Conservative forces and central forces lead to conservation of *energy* and *angular momentum*, respectively. In this section we focus on developing the theory, with a few very simple examples, but then apply this to more sophisticated examples in sections 5 and 6.

## 4.1 Planar motion in polar coordinates

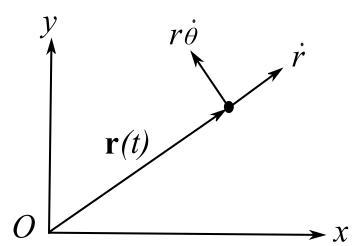
Motion in a plane is sometimes conveniently described using polar coordinates. Recall that Cartesian coordinates $(x, y)$ are related to polar coordinates $(r, \theta)$ by

$$x = r\cos\theta\ , \qquad y = r\sin\theta\ . \tag{4.1}$$

See Figure 13a. The coordinate $r = \sqrt{x^2+y^2} \geq 0$ is simply the Euclidean distance of the point $(x, y)$ from the origin $O$. On the complement of the origin we have $\tan\theta = \frac{y}{x}$, where $\theta \in [0, 2\pi)$, and the direction of increasing $\theta$ is anticlockwise.



(a) Cartesian and polar coordinates.



(b) Velocity in polar coordinates.

Figure 13

We next introduce the two unit vectors

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j}\ , \qquad \mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}\ . \tag{4.2}$$

These should be thought of as direction vectors at a point with polar coordinates $(r, \theta)$, $r \neq 0$, as in Figure 13a. $\mathbf{e}_r$ is a unit vector in the direction of increasing $r$ (at fixed $\theta$), while $\mathbf{e}_\theta$ is a unit vector in the direction of increasing $\theta$ (at fixed $r$). We also have $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$, so that at every point in the plane (apart from the origin) $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ form an orthonormal basis. However, unlike $\{\mathbf{i}, \mathbf{j}\}$ the directions of the vectors $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are not fixed, but depend on $\theta$.

In this basis the position of a particle is simply $\mathbf{r} = (x, y) = r\,\mathbf{e}_r$. For a time-dependent trajectory $\mathbf{r}(t)$ we then compute

$$\dot{\mathbf{r}} = \dot{r}\,\mathbf{e}_r + r\,\dot{\mathbf{e}}_r\ . \tag{4.3}$$

But from (4.2) we have

$$\begin{aligned}\dot{\mathbf{e}}_r &= -\dot{\theta}\sin\theta\,\mathbf{i} + \dot{\theta}\cos\theta\,\mathbf{j} \;=\; \dot{\theta}\,\mathbf{e}_\theta \;, \\ \dot{\mathbf{e}}_\theta &= -\dot{\theta}\cos\theta\,\mathbf{i} - \dot{\theta}\sin\theta\,\mathbf{j} \;=\; -\dot{\theta}\,\mathbf{e}_r \;, \end{aligned} \tag{4.4}$$

and hence

$$\dot{\mathbf{r}} \;=\; \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta \;. \tag{4.5}$$

The second term has arisen because the basis we used is itself time-dependent, specifically due to the time-dependence of $\theta = \theta(t)$. The quantity $\dot{\theta}$ is called the *angular velocity*. Equation (4.5) expresses velocity $\dot{\mathbf{r}}$ in polar coordinates – see Figure 13b. We may find a similar expression for acceleration by taking another time derivative, using (4.4):

$$\begin{aligned}\ddot{\mathbf{r}} &= \ddot{r}\,\mathbf{e}_r + \dot{r}\,\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\,\mathbf{e}_\theta + r\ddot{\theta}\,\mathbf{e}_\theta + r\dot{\theta}\,\dot{\mathbf{e}}_\theta \;, \\ &= (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\mathbf{e}_\theta \;, \\ &= (\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\,\mathbf{e}_\theta \;, \end{aligned} \tag{4.6}$$

Here in the last line we've written $2\dot{r}\dot{\theta} + r\ddot{\theta} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})$.

**Example** (Uniform circular motion): Consider a particle moving in a circle of radius $R$, centre the origin, at constant speed $v$. Since $r = R = \text{constant}$ we have $\dot{r} = 0$. Thus from (4.5) its velocity is

$$\dot{\mathbf{r}} \;=\; R\,\dot{\theta}\,\mathbf{e}_\theta \;. \tag{4.7}$$

This is tangent to the circle. The particle's speed is $v = |\dot{\mathbf{r}}|$, which implies $v = R|\dot{\theta}|$, and hence the angular speed $|\dot{\theta}| = \frac{v}{R}$ is constant. Since $\dot{\theta}$ is constant, $\ddot{\theta} = 0$, and similarly since $\dot{r} = 0$ we also have $\ddot{r} = 0$. Thus from (4.6) the acceleration is

$$\ddot{\mathbf{r}} \;=\; -R\,\dot{\theta}^2\,\mathbf{e}_r \;=\; -\frac{v^2}{R}\,\mathbf{e}_r \;. \tag{4.8}$$

We conclude that the acceleration in uniform circular motion has magnitude $v^2/R$, and is directed towards the centre of the circle $O$. Newton's second law implies that in order to generate this acceleration we need a force of magnitude $F = mv^2/R = mR\,\dot{\theta}^2$ directed towards the origin – this is called the *centripetal force*. ■

## 4.2 Conservative forces

In section 3.1 we saw that for motion in one dimension and forces $F = F(x)$ there is a conserved energy. In three dimensions this is no longer necessarily the case: we need an additional constraint on $\mathbf{F} = \mathbf{F}(\mathbf{r})$ in order for energy to be conserved. Even before looking at the details one might have anticipated this: energy is a scalar quantity, and without any further input there is no natural way to construct a scalar from the vector $\mathbf{F}$, analogously to what we did in one dimension.

**Definition** The *kinetic energy* of a particle is $T = \frac{1}{2}m|\dot{\mathbf{r}}|^2$, where $\mathbf{r}(t)$ is the particle's position in an inertial frame.

We then have the following important result:

**Conservation of Energy Theorem** The quantity

$$E = T + V = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r}) , \tag{4.9}$$

is *conserved* if the force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ takes the form

$$\mathbf{F} = -\nabla V . \tag{4.10}$$

That is, in Cartesian coordinates $\mathbf{F} = (-\partial_x V, -\partial_y V, -\partial_z V)$.

**Proof**: Suppose that the force takes the form (4.10). Using the chain rule we compute

$$\begin{aligned} \dot{E} &= m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} + \nabla V \cdot \dot{\mathbf{r}} \\ &= (m\ddot{\mathbf{r}} - \mathbf{F}) \cdot \dot{\mathbf{r}} = 0 , \end{aligned} \tag{4.11}$$

where the last step uses Newton's second law. ■

To understand where the condition (4.10) really comes from, it is useful to first generalize the notion of work to three dimensions:

**Definition** The *work done* by a force $\mathbf{F}$ in moving a particle from $\mathbf{r}_1$ to $\mathbf{r}_2$ along a curve $C$ is

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} . \tag{4.12}$$

The distinction with the corresponding definition in one dimension (3.4) is that in higher dimensions the line integral (4.12) depends on the precise curve $C$, and not just on its endpoints $\mathbf{r}_1$, $\mathbf{r}_2$. If we now suppose that $\mathbf{r}(t)$ is the trajectory of a particle satisfying Newton's second law, starting at position $\mathbf{r}_1 = \mathbf{r}(t_1)$ and ending at $\mathbf{r}_2 = \mathbf{r}(t_2)$, then we may write

$$W = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}} \, \mathrm{d}t = m \int_{t_1}^{t_2} \ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} \, \mathrm{d}t = \frac{1}{2}m \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}|\dot{\mathbf{r}}|^2 \, \mathrm{d}t = T(t_2) - T(t_1) . \tag{4.13}$$

Thus, as in one dimension, the work done by the force is the change in kinetic energy.

Suppose now that the total energy $E$ given by (4.9) is conserved. This means that $E = T(t_1) + V(\mathbf{r}_1) = T(t_2) + V(\mathbf{r}_2)$, and hence (4.13) implies that

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = V(\mathbf{r}_1) - V(\mathbf{r}_2) . \tag{4.14}$$

The right hand side manifestly depends only on the endpoints $\mathbf{r}_1$, $\mathbf{r}_2$ of the curve $C$, and we have thus shown that if energy is conserved then the work done is *independent* of the choice of curve $C$

connecting $\mathbf{r}_1$ to $\mathbf{r}_2$. In the Prelims Multivariable Calculus course you prove that if this is true for *all* curves $C$ then $\mathbf{F}$ takes the form (4.10).[12]

**Definition** A force $\mathbf{F} = \mathbf{F}(\mathbf{r})$ is said to be *conservative* if there exists a *potential energy* function $V = V(\mathbf{r})$ such that

$$\mathbf{F} = -\nabla V \ . \tag{4.15}$$

Note that as in one dimension the potential $V$ is only defined up to an additive constant.

**Examples**:

(i) Any *constant* force $\mathbf{F}_{\text{const}}$ is conservative, with potential $V(\mathbf{r}) = -\mathbf{F}_{\text{const}} \cdot \mathbf{r}$. An important example is gravity: for $\mathbf{F} = -mg\,\mathbf{k}$ the corresponding potential function is simply $V(\mathbf{r}) = mg\,\mathbf{k} \cdot \mathbf{r} = mgz$.

(ii) In section 6.1 we'll show that any force of the form $\mathbf{F} = F(|\mathbf{r}|)\,\mathbf{e}_r$ is conservative, where $\mathbf{e}_r = \mathbf{r}/|\mathbf{r}|$. These also play a particularly important role in Dynamics.

Conservative forces enjoy the following equivalent definitions:

**Theorem** (From Prelims Multivariable Calculus) Let $\mathbf{F} : S \to \mathbb{R}^3$ be a vector field, where the domain $S \subset \mathbb{R}^3$ is open and path connected. Then the following three statements are equivalent:

1. $\mathbf{F}$ is conservative, *i.e.* there exists a potential $V : S \to \mathbb{R}$ such that $\mathbf{F} = -\nabla V$.

2. Given any two points $\mathbf{r}_1$, $\mathbf{r}_2$ in $S$, and any curve $C$ in $S$ starting at $\mathbf{r}_1$ and ending at $\mathbf{r}_2$, then the integral $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is independent of the choice of $C$.

3. For any simple closed curve $C$ in $S$ we have $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 0$.

It is also shown in Multivariable Calculus that conservative forces satisfy $\nabla \wedge \mathbf{F} = \mathbf{0}$, although we won't need this fact.

## 4.3 Central forces and angular momentum

Another important concept is that of a *central force*:

**Definition** A force that is always directed along the line joining a particle to a fixed position in an inertial frame is called a *central force*. It is usually convenient to choose this point as the origin of the frame, meaning that

$$\mathbf{F} \ \propto \ \mathbf{r} \ , \tag{4.16}$$

where $\mathbf{r}$ is the position vector of the particle, measured from the origin $O$.

[12] See the Theorem below. Note also that *if* $\mathbf{F}$ takes the form (4.10), the second equality in (4.14) is just the Fundamental Theorem of Calculus.

The importance of central forces is that they always lead to an associated conserved vector quantity.

**Definition** The *angular momentum* $\mathbf{L} = \mathbf{L}_P$ of a particle about a point $P$ in an inertial frame is the *moment* of linear momentum $\mathbf{p} = m\dot{\mathbf{r}}$ *about* $P$. That is,

$$\mathbf{L}_P \equiv (\mathbf{r} - \mathbf{x}) \wedge m\dot{\mathbf{r}} = (\mathbf{r} - \mathbf{x}) \wedge \mathbf{p} . \tag{4.17}$$

Here $\mathbf{x}$ is the position vector of the point $P$, while $\mathbf{r}$ is the position of the particle (both measured from the origin $O$). Notice that $\dot{\mathbf{r}}$ is the velocity of the particle in the inertial frame, *not* the velocity relative to $P$ (which could in principle be moving, $\mathbf{x} = \mathbf{x}(t)$).

This definition makes it clear that the angular momentum depends on the point $P$. However, for central forces there is a natural choice for $P$, namely $P = O$, the centre of the force.

**Proposition** If a particle is acted on by a central force with centre $O$ then the angular momentum $\mathbf{L} = \mathbf{L}_O$ is conserved, and the path of the particle lies entirely in a fixed plane through $O$. That is, the motion is *planar*.

**Proof**: We compute

$$\dot{\mathbf{L}} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r} \wedge m\dot{\mathbf{r}}) = \dot{\mathbf{r}} \wedge m\dot{\mathbf{r}} + \mathbf{r} \wedge m\ddot{\mathbf{r}} = \mathbf{r} \wedge m\ddot{\mathbf{r}} . \tag{4.18}$$

The last equality follows since $\mathbf{a} \wedge \mathbf{a} = \mathbf{0}$ for any vector $\mathbf{a}$. Newton's second law then gives

$$\dot{\mathbf{L}} = \mathbf{r} \wedge m\ddot{\mathbf{r}} = \mathbf{r} \wedge \mathbf{F} = \mathbf{0} , \tag{4.19}$$

where the last equality follows since $\mathbf{F} \propto \mathbf{r}$ for a central force. Thus $\mathbf{L}$ is constant. In the special case that $\mathbf{L} = \mathbf{r} \wedge m\dot{\mathbf{r}} = \mathbf{0}$ the position and velocity must be parallel ($\mathbf{a} \wedge \mathbf{b} = \mathbf{0}$ implies $\mathbf{a}$ and $\mathbf{b}$ are parallel, where we include this to mean that one or both are zero). Thus the particle either moves in a straight line through the origin, or in the special case that $\dot{\mathbf{r}} = \mathbf{0}$ sits at a fixed point. More generally if $\mathbf{L} \neq \mathbf{0}$ then notice that $\mathbf{L} \cdot \mathbf{r} = 0 = \mathbf{L} \cdot \dot{\mathbf{r}}$ both follow from $\mathbf{L} = \mathbf{r} \wedge m\dot{\mathbf{r}}$, meaning that $\mathbf{L}$ is always perpendicular to both the position of the particle and its velocity. This means that the motion is confined to the plane through $O$ with normal vector $\mathbf{L}$ – see Figure 14. ■

Suppose that $\mathbf{L} = \mathbf{L}_O$ is conserved, as in the last Proposition. In particular the direction of $\mathbf{L}$ is constant, and we may choose this as the $z$ direction, so that $\mathbf{L}_O \propto \mathbf{k}$. Introducing polar coordinates for the planar motion in the $(x, y)$ plane, we compute

$$\begin{aligned} \mathbf{L} = \mathbf{r} \wedge m\dot{\mathbf{r}} &= r\,\mathbf{e}_r \wedge m\left(\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta\right) , \\ &= mr^2\dot{\theta}\,\mathbf{k} , \end{aligned} \tag{4.20}$$

where we have used (4.5) in the first line, and $\mathbf{k} = \mathbf{i} \wedge \mathbf{j} = \mathbf{e}_r \wedge \mathbf{e}_\theta$ in the last step. This proves the following result, which will be important later:

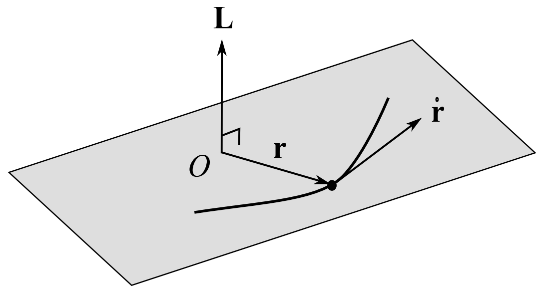



Figure 14: The planar motion of a particle acted on by a central force, with centre $O$. The conserved angular momentm $\mathbf{L}$ is normal to the plane of motion through $O$.

**Proposition** If angular momentum $\mathbf{L}$ is conserved, then the quantity

$$h \equiv r^2\dot{\theta} \equiv \textit{specific} \text{ angular momentum} \tag{4.21}$$

is conserved, where $(r, \theta)$ are polar coordinates in the plane of motion. Note from (4.20) that $|\mathbf{L}| = m|h|$, so that $h$ is also the *angular momentum per unit mass*.

For completeness we conclude this section with a definition and brief discussion of *torque*, although in practice we won't meet this again until section 7.

**Definition** The *torque* $\boldsymbol{\tau} = \boldsymbol{\tau}_P$ of a force $\mathbf{F}$, about a point $P$ with position vector $\mathbf{x}$, acting on a particle with position vector $\mathbf{r}$ is

$$\boldsymbol{\tau}_P \equiv (\mathbf{r} - \mathbf{x}) \wedge \mathbf{F} . \tag{4.22}$$

In other words, the torque is the *moment* of the force about $P$. The direction of $\boldsymbol{\tau}_P$ is normal to the plane containing $\mathbf{r} - \mathbf{x}$ and $\mathbf{F}$, and may be regarded as the *axis* about which the force tends to rotate the particle about $P$.

If $P$ is a fixed point in the inertial frame, so that $\mathbf{x} =$ constant, then using (4.17) and Newton's second law we have

$$\dot{\mathbf{L}}_P = (\mathbf{r} - \mathbf{x}) \wedge m\ddot{\mathbf{r}} = (\mathbf{r} - \mathbf{x}) \wedge \mathbf{F} = \boldsymbol{\tau}_P , \tag{4.23}$$

and the torque is the rate of change of angular momentum. This can be compared with Newton's second law itself, written in the form $\dot{\mathbf{p}} = \mathbf{F}$, which says that the force is the rate of change of linear momentum. The definition (4.22) leads to another way to characterize central forces:

**Proposition** A force is a central force about $P$ if and only if the torque about $P$ is zero, or equivalently $\mathbf{L}_P$ is conserved.

**Proof**: The torque (4.22) is zero if and only if $\mathbf{F} \propto (\mathbf{r} - \mathbf{x})$, which means $\mathbf{F}$ is a central force about $P$. On the other hand from (4.23) the torque about $P$ is zero if and only if $\dot{\mathbf{L}}_P = \mathbf{0}$. ■

# 5 Constrained systems

In this section we consider *constrained* dynamical systems: think of masses attached to light rods, beads threaded on smooth wires, marbles rolling in smooth dishes, *etc.* The dynamics happens in $\mathbb{R}^3$, but the constraints effectively reduce the motion to a one-dimensional or two-dimensional dynamical system.

## 5.1 Constraint forces

If a particle is going to be constrained to move on a particular curve or surface in $\mathbb{R}^3$, there must be some kind of force ensuring this. These types of forces are a little different to those we have considered so far, but they may be included in Newton's second law just as easily. In this course we shall make the following assumption about these constraint forces:

**Assumption**: The constraint force $\mathbf{N}$ is always *perpendicular* to the constraint space.

We have used the letter "N" because "perpendicular" is also sometimes called "normal", and such constraint forces are similarly also referred to as *normal reaction* forces. Since by definition the velocity of the particle $\dot{\mathbf{r}}$ is always *tangent* to the constraint space, we have

$$\mathbf{N} \cdot \dot{\mathbf{r}} \;=\; 0 \,. \tag{5.1}$$

This is a simple geometric condition, but what does this Assumption mean *physically*? The *work done* by the force $\mathbf{N}$ when the particle moves along a curve $C$ in the constraint space is (from the definition (4.12))

$$W \;=\; \int_C \mathbf{N} \cdot \mathrm{d}\mathbf{r} \;=\; \int \mathbf{N} \cdot \dot{\mathbf{r}} \, \mathrm{d}t \;=\; 0 \,, \tag{5.2}$$

where the last step uses (5.1). Thus such constraint forces *do no work* during the constrained motion of the particle. Another way to think about this is that there is no component of the constraint force *tangent* to the constraint space. Actually any reaction force tangent to the constraint space would be interpreted as some kind of *friction* force, opposing motion along the wire, dish, or whatever the constraint space is. Thus an equivalent Assumption is to say that that the constraint space is *smooth*, or *frictionless*: the implication is that $\mathbf{N}$ is perpendicular to the constraint space, and hence does no work.

If we consider a particle of mass $m$, acted on by a force $\mathbf{F}_0$, that is then further constrained to move on a smooth constraint space, Newton's second law simply reads

$$m\ddot{\mathbf{r}} \;=\; \mathbf{F} \;=\; \mathbf{F}_0 + \mathbf{N} \,, \tag{5.3}$$

where $\mathbf{N}$ is the normal reaction/constraint force. We have the following important result:

**Conservation of Energy Theorem** (Constrained motion) Suppose that the force $\mathbf{F}_0 = -\nabla V$ is conservative, with potential $V = V(\mathbf{r})$. Then the total energy $E = T + V$ is conserved in the *constrained* motion of the particle.

**Proof**: We simply compute

$$\begin{aligned}\dot{E} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r})\right) &&= m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} + \nabla V\cdot\dot{\mathbf{r}} \\ & &&= (m\ddot{\mathbf{r}} - \mathbf{F}_0)\cdot\dot{\mathbf{r}} \\ & &&= (m\ddot{\mathbf{r}} - \mathbf{F}_0 - \mathbf{N})\cdot\dot{\mathbf{r}} \ = \ 0\ .\end{aligned} \tag{5.4}$$

The first few steps are identical to the proof of conservation of energy in the unconstrained case. In going to the last line we have used (5.1), and the last equality is Newton's second law (5.3). ■

Let's see all of this in some examples.

## 5.2 The simple pendulum

Perhaps the simplest interesting example of constrained motion is the *simple pendulum*. This consists of a mass $m$ fixed to the end of a light (*i.e.* negligible mass) rod of length $l$. The other end of the rod is hinged smoothly at a point $O$ and is free to swing in a vertical plane under gravity.

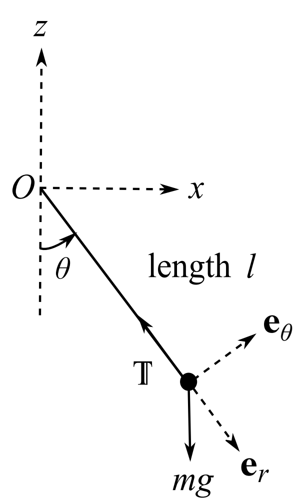



Figure 15: A simple pendulum.

The set up is shown in Figure 15. The effect of the rod is to constrain the mass $m$ to move on a circle of radius $l$ in the $(z, x)$ plane, centred on the pivot point $O$. The constraint space in this case is hence a circle. The constraint force for the motion is the *tension* $\mathbb{T}$ in the rod. We denote the angle that the rod makes with the downward vertical by $\theta$. Notice that this is the only degree of freedom in the problem, parametrizing where the mass $m$ is on the circle, so we expect to find an ODE for $\theta(t)$ from Newton's laws.

Given that the motion will lie on a circle, it is useful to introduce polar coordinates in the $(z, x)$ plane: $z = -l\cos\theta$, $x = l\sin\theta$. The corresponding unit vectors are

$$\mathbf{e}_r \;=\; -\cos\theta\,\mathbf{k} + \sin\theta\,\mathbf{i}\,, \qquad \mathbf{e}_\theta \;=\; \sin\theta\,\mathbf{k} + \cos\theta\,\mathbf{i}\,. \tag{5.5}$$

See Figure 15. Although these are slightly different to the polar coordinates in the $(x, y)$ plane in Figure 13a, the essential point is that as in (4.4) we again have $\dot{\mathbf{e}}_r = \dot\theta\,\mathbf{e}_\theta$, $\dot{\mathbf{e}}_\theta = -\dot\theta\,\mathbf{e}_r$. It follows that the velocity and acceleration are again given by (4.5) and (4.6), respectively, where $\mathbf{r} = (z, x)$.

The forces acting on the mass $m$ are gravity and the constraint force: in the notation of section 5.1 we have

$$\mathbf{F}_0 \;=\; -mg\,\mathbf{k}\,, \qquad \mathbf{N} \;=\; -\mathbb{T}\,\mathbf{e}_r\,, \tag{5.6}$$

where the total force acting is $\mathbf{F} = \mathbf{F}_0 + \mathbf{N}$. Notice in particular that the constraint force $\mathbf{N}$ acts in the radial direction, and is thus always perpendicular to the constrained motion in a circle: this follows from our general discussion in section 5.1, and the fact that the rod is "hinged smoothly" at $O$.

Newton's second law (5.3) is a vector equation, and we may conveniently pick out different components of it by taking dot products with the linearly independent vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$. Since $\mathbf{k} = -\cos\theta\,\mathbf{e}_r + \sin\theta\,\mathbf{e}_\theta$, taking the dot product of Newton's second law (5.3) with $\mathbf{e}_\theta$ gives

$$m\ddot{\mathbf{r}}\cdot\mathbf{e}_\theta \;=\; \mathbf{F}\cdot\mathbf{e}_\theta \;=\; -mg\,\sin\theta\,. \tag{5.7}$$

From (4.6) we have $\ddot{\mathbf{r}}\cdot\mathbf{e}_\theta = 2\dot r\dot\theta + r\ddot\theta$. However, here $r = l$ is constant, so that (5.7) reads $ml\ddot\theta = -mg\sin\theta$, which rearranges to

$$\ddot\theta \;=\; -\frac{g}{l}\sin\theta\,. \tag{5.8}$$

This is the equation of motion for the simple pendulum: a second order ODE for $\theta(t)$. Of course, this is only "half" of Newton's second law. To obtain the remaining equation we take the dot product of (5.3) with $\mathbf{e}_r$:

$$m\ddot{\mathbf{r}}\cdot\mathbf{e}_r \;=\; \mathbf{F}\cdot\mathbf{e}_r \;=\; mg\,\cos\theta - \mathbb{T}\,. \tag{5.9}$$

On the other hand from (4.6) we have $\ddot{\mathbf{r}}\cdot\mathbf{e}_r = \ddot r - r\dot\theta^2 = -l\dot\theta^2$. Thus (5.9) rearranges to

$$\mathbb{T} \;=\; ml\dot\theta^2 + mg\cos\theta\,. \tag{5.10}$$

This says that the tension $\mathbb{T}$ balances the component of the weight along the rod $mg\cos\theta$, and the centripetal force $ml\dot\theta^2$ for circular motion about the origin $O$.

We cannot solve the equation of motion (5.8) in closed form, as simple as it looks. However, we have our dynamics toolbox to apply: let's look at the *equilibrium configurations*, and *conservation of energy*.

**Equilibria**: Notice there are two equilibrium configurations, where the right hand side of (5.8) is zero: $\theta = 0$ and $\theta = \pi$. The former has the pendulum hanging down vertically, and for small oscillations (*i.e.* small $\theta$) we may approximate $\sin\theta \simeq \theta$. In this linearized approximation (5.8) becomes

$$\ddot{\theta} \;=\; -\omega^2\theta \;, \qquad \text{where} \qquad \omega^2 \;=\; \frac{g}{l} > 0 \;. \tag{5.11}$$

Thus, as is intuitively obvious, $\theta = 0$ is a *stable equilibrium*, in the terminology of section 3.3. For small oscillations about this point the pendulum executes simple harmonic motion with angular frequency $\omega$, and hence period

$$T \;=\; 2\pi\sqrt{\frac{l}{g}} \;. \tag{5.12}$$

Notice that $\sqrt{l/g}$ indeed has dimensions of time, and that in fact this is the only way we can construct such a quantity from the variables in the problem. Thus $T$ had to be a dimensionless number times $\sqrt{l/g}$.

The second equilibrium position, $\theta = \pi$, has the rod precariously balanced above the pivot point $O$. Setting $\theta = \pi + \xi(t)$, with $\xi(t)$ small, we may now approximate $\sin\theta = \sin(\pi+\xi) \simeq -\sin\xi \simeq -\xi$. The linearized equation of motion obtained from (5.8) thus reads

$$\ddot{\xi} \;=\; -\frac{g}{l}(-\xi) \;=\; \frac{g}{l}\xi \;. \tag{5.13}$$

The general solution is $\xi(t) = C\,\mathrm{e}^{\sqrt{g/l}\,t} + D\,\mathrm{e}^{-\sqrt{g/l}\,t}$, and the equilibrium is unstable.

**Conservation of energy**: The Conservation of Energy Theorem at the end of section 5.1 guarantees that the total energy of the mass is conserved: the gravitational force $\mathbf{F}_0 = -mg\,\mathbf{k}$ is conservative, with potential $V(\mathbf{r}) = V(x,y,z) = mgz$. The total energy is

$$E \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(\mathbf{r}) \;=\; \frac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta \;. \tag{5.14}$$

Here in the second equality we have substituted $\dot{\mathbf{r}} = l\dot{\theta}\,\mathbf{e}_\theta$ for circular motion, and $z = -l\cos\theta$. Let's check explicitly that $E$ is indeed conserved:

$$\dot{E} \;=\; ml^2\ddot{\theta}\dot{\theta} + mgl\sin\theta\,\dot{\theta} \;=\; ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right) \;. \tag{5.15}$$

We thus see that $\dot{E} = 0$, provided the equation of motion (5.8) holds.

As in section 3.2, we may view (5.14) as a *first order* ODE for $\theta(t)$, and integrate it. Rerranging we have

$$\dot{\theta}^2 \;=\; \frac{2E}{ml^2} + \frac{2g}{l}\cos\theta \;, \tag{5.16}$$

which integrates to

$$t \;=\; \pm\int \frac{\mathrm{d}\theta}{\sqrt{2E/ml^2 + 2(g/l)\cos\theta}} \;. \tag{5.17}$$

If we assume that the pendulum starts at $\theta = 0$ at time $t = 0$, and reaches a maximum angle of $\theta_0 > 0$ in its swing, then we may compute the period of the swing:

$$T = 4\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{2E/ml^2 + 2(g/l)\cos\theta}} = 4\sqrt{\frac{l}{g}}\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\sqrt{2\cos\theta - 2\cos\theta_0}} . \tag{5.18}$$

Here we have noted that at the top of the swing $\dot{\theta} = 0$, and hence from (5.16) $\cos\theta_0 = -E/mgl$. The factor of 4 in (5.18) arises because the integral from 0 to $\theta_0$ is only a quarter of one period. Compare to the result for small oscillations (5.12). One can derive this from the general formula (5.18) by making the approximation $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$ in the integral. More generally the integral in (5.18) is an *elliptic integral*. We also see that the period $T$ is a dimensionless number times $\sqrt{l/g}$, where the dimensionless number in general depends on the initial conditions (via the conserved energy $E$).

Finally, notice that we have tacitly assumed that $|E| \leq mgl$ in the above discussion. From (5.14) we have $E \geq -mgl$, with equality for the stable equilibrium at $\theta = 0$. However, if $E > mgl$ then $\cos\theta_0 = -E/mgl$ has no solution, and hence $\dot{\theta}$ is never zero. In this case the system has so much energy that the pendulum swings over the top of the pivot point.

## 5.3 Motion on a surface under gravity

Consider a mass $m$ moving under gravity on a smooth surface. For example, this might model a marble rolling in a dish, or a bicycle freewheeling down a hill. The gravitational force is, as usual, $\mathbf{F}_0 = -mg\,\mathbf{k}$, and this is conservative with potential $V(\mathbf{r}) = mgz$. The fact that the surface is smooth means that the constraint force is perpendicular to the surface.

Mathematically, there are different ways in which we can specify a surface in $\mathbb{R}^3$ (see the last part of the Geometry course). For example, we can define a surface as the zero set of some suitable function $f : \mathbb{R}^3 \to \mathbb{R}$. A normal vector to the surface is $\nabla f$, and the constraint force will be proportional to this. Rather than trying to describe the general situation, here we'll focus on a *surface of revolution*. This may be specified as

$$f(r,\theta,z) \equiv z - H(r) = 0 , \tag{5.19}$$

where $(r,\theta,z)$ are *cylindrical polar coordinates*. Recall this means that the Cartesian $(x,y)$ coordinates are given by $x = r\cos\theta$, $y = r\sin\theta$. The defining equation (5.19) specifies the height $z$ as a function $z = H(r)$ of the radial distance $r$ in the $(x,y)$ plane – see Figure 16. Since this is independent of $\theta$ the resulting surface will be invariant under rotation about the $z$ axis, which rotates the $\theta$ coordinate. This also implies that $\mathbf{e}_\theta$ is *tangent* to the surface at every point, and hence in particular we have $\mathbf{N}\cdot\mathbf{e}_\theta = 0$.

The position vector of the particle moving on the surface is

$$\mathbf{r} = r\,\mathbf{e}_r + z\,\mathbf{e}_z , \tag{5.20}$$

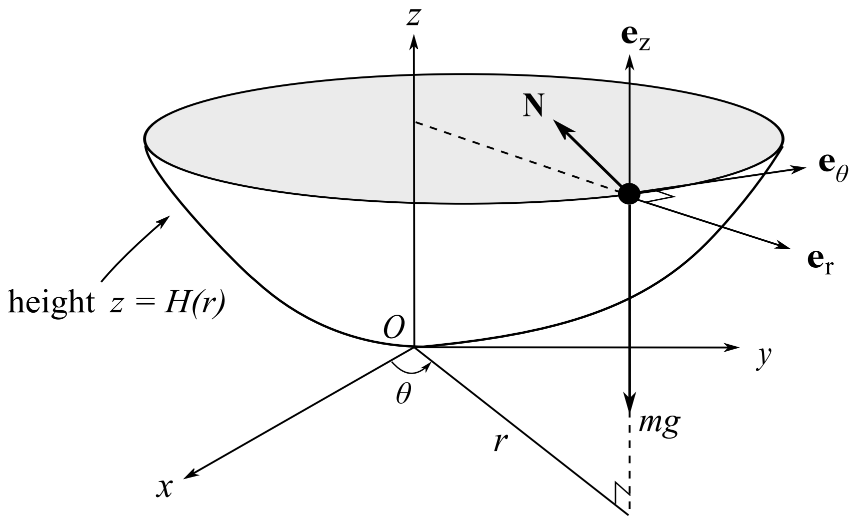



Figure 16: Particle moving under gravity on a surface of revolution.

where $\mathbf{e}_z = \mathbf{k}$. Newton's second law (5.3) thus reads

$$m\ddot{\mathbf{r}} \;=\; \mathbf{F} \;=\; -mg\,\mathbf{e}_z + \mathbf{N}\ . \tag{5.21}$$

Using (4.6) we may write the acceleration $\ddot{\mathbf{r}}$ in cylindrical polar coordinates, so that (5.21) reads

$$m\left[\left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right)\mathbf{e}_\theta + \ddot{z}\,\mathbf{e}_z\right] \;=\; -mg\,\mathbf{e}_z + \mathbf{N}\ . \tag{5.22}$$

Notice that every term in (5.22) is orthogonal to $\mathbf{e}_\theta$, apart from the term proportional to $\mathbf{e}_\theta$. Thus we immediately deduce

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) \;=\; 0 \qquad \Longrightarrow \qquad r^2\dot{\theta} \;\equiv\; h \;=\; \text{constant}\ . \tag{5.23}$$

From the general discussion in section 4.3 we see this has something to do with conservation of angular momentum. However, in that section we also showed that the angular momentum $\mathbf{L}_P$ about a point $P$ is conserved if and only if the force acting is a central force about $P$, *i.e.* the total force $\mathbf{F}$ is always directed towards $P$. This clearly isn't true in general for motion on a surface of revolution. To see what's going on, let us compute the angular momentum about the origin $O$:

$$\begin{aligned}\mathbf{L} \;=\; \mathbf{L}_O \;=\; \mathbf{r}\wedge m\dot{\mathbf{r}} \;&=\; m\,(r\,\mathbf{e}_r + z\,\mathbf{e}_z)\wedge\left(\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z\right)\\ &=\; mr^2\dot{\theta}\,\mathbf{e}_z + m(z\dot{r} - r\dot{z})\,\mathbf{e}_\theta - mzr\dot{\theta}\,\mathbf{e}_r\ ,\end{aligned} \tag{5.24}$$

where we have used $\mathbf{e}_r\wedge\mathbf{e}_\theta = \mathbf{e}_z$, $\mathbf{e}_\theta\wedge\mathbf{e}_z = \mathbf{e}_r$, $\mathbf{e}_z\wedge\mathbf{e}_r = \mathbf{e}_\theta$, which are easily checked from (4.2). We thus have

$$\mathbf{L}\cdot\mathbf{e}_z \;=\; mr^2\dot{\theta} \;=\; mh\ . \tag{5.25}$$

We now see that (5.23) says that the *component of angular momentum in the direction of the axis of symmetry* $\mathbf{e}_z$ *is conserved*. In fact this is directly related to the rotational symmetry about this axis, although an explanation of this will have to wait for the course B7.1 on Classical Mechanics.[13]

Let's go back to Newton's second law (5.22). This is a vector differential equation, and we have so far taken the dot product with $\mathbf{e}_\theta$ to find the conserved quantity $h$ in (5.23). There must be two more scalar equations, and obviously we may obtain these by taking dot products with $\mathbf{e}_r$ and $\mathbf{e}_z$. However, we should be a bit smarter and think about what we're trying to do. The motion of the particle is determined by finding $r(t)$, $\theta(t)$ and $z(t)$. Firstly, $z(t)$ is fixed to be $z(t) = H(r(t))$ by the constraint (5.19). Secondly, the equation (5.23) determines $\dot{\theta} = h/r^2(t)$, which may be integrated to find $\theta(t)$, once we know $r(t)$. Thus at this stage we really only have one degree of freedom in the problem, namely $r(t)$. One linear combination of the remaining equations in (5.22) should thus be an equation of motion for $r(t)$. The other combination simply determines the constraint force $\mathbf{N}$, *c.f.* the simple pendulum, where equation (5.9) determined the tension in the pendulum. If we want to obtain the equation of motion for $r(t)$ directly, a nice geometric way to do this is to take the dot product of (5.22) with another *tangent vector* to the surface: that way $\mathbf{N}$ will immediately drop out. Since $f(r, \theta, z) = z - H(r) = 0$ defines the surface, a normal vector is[14]

$$\mathbf{n} \;=\; \nabla f \;=\; \mathbf{e}_z - H'(r)\mathbf{e}_r \; . \tag{5.26}$$

The constraint force $\mathbf{N}$ is proportional to $\mathbf{n}$. We have already used that $\mathbf{e}_\theta$ is tangent to the surface, and from (5.26) another independent tangent vector is

$$\mathbf{t} \;=\; H'(r)\mathbf{e}_z + \mathbf{e}_r \; . \tag{5.27}$$

Clearly $\mathbf{t}\cdot\mathbf{n} = 0$. Thus taking the dot product of Newton's second law (5.22) with $\mathbf{t}$ gives (cancelling an overall factor of the mass $m$)

$$\left(\ddot{r} - r\dot{\theta}^2\right) + H'(r)\,\ddot{z} \;=\; -g\,H'(r) \; . \tag{5.28}$$

On the other hand from (5.23) and the defining equation (5.19) we may substitute

$$\dot{\theta} \;=\; \frac{h}{r^2} \, , \qquad z \;=\; H(r) \, , \tag{5.29}$$

and from the chain rule

$$\dot{z} \;=\; H'(r)\,\dot{r} \, , \qquad \ddot{z} \;=\; H''(r)\,\dot{r}^2 + H'(r)\,\ddot{r} \; . \tag{5.30}$$

Substituting these into (5.28) hence gives

$$\left[1 + \left(H'(r)\right)^2\right]\ddot{r} + H'(r)H''(r)\,\dot{r}^2 - \frac{h^2}{r^3} \;=\; -g\,H'(r) \; . \tag{5.31}$$

[13]** *Noether's Theorem*, due to Emmy Noether, relates any continuous symmetry to a corresponding conserved quantity. In fact all the conserved quantities in this course arise in this way.

[14]The gradient in cylindrical polar coordinates is $\nabla f = \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z$. See the Multivariable Calculus course.

This is a second order ODE for $r(t)$, as we expected to find. Given $H(r)$, in principle one can try to solve this equation.

However, from the end of section 5.1 we also know that there is a conserved energy for this problem. This will lead to a *first order* equation, and moreover the second order equation (5.31) should be *implied* by this first order equation. Let's see that this is indeed the case. From conservation of energy we know that

$$E = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + mgz = \text{constant} . \tag{5.32}$$

In cylindrical polars we compute

$$|\dot{\mathbf{r}}|^2 = |\dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta + \dot{z}\,\mathbf{e}_z|^2 = \dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2 . \tag{5.33}$$

Substituting using (5.29) into (5.32) we hence have

$$E = \frac{1}{2}m\left[\dot{r}^2 + \frac{h^2}{r^2} + \left(H'(r)\right)^2\dot{r}^2\right] + mg\,H(r) . \tag{5.34}$$

Equation (5.34) is the expected first order ODE for $r(t)$. It's straightforward enough to take $\mathrm{d}/\mathrm{d}t$ of the right hand side of (5.34) and check (after cancelling an overall factor of $\dot{r}$) this indeed gives the equation (5.31). In the other direction, equation (5.34) may in principle be integrated, given $H(r)$, although typically one won't be able to do the integral explicitly. Nevertheless, as we have seen before one can often use conservation of energy to deduce various qualitative and quantitative features of the motion. Let's see this in a concrete example, with a specific choice of the surface of revolution (*i.e.* choice of $H(r)$).

**Example** (Motion on a paraboloid): A particle moves under gravity on the smooth inside surface of the paraboloid $z = r^2/4a$. Initially it is at a height $z = a$ and is projected horizontally with speed $v$ along the surface of the paraboloid. Show that the particle moves between two heights in the subsequent motion, and find them.

**Solution**: We have $H(r) = r^2/4a$. We begin by substituting the initial conditions into the conserved specific angular momentum $h = r^2\dot{\theta}$. At $t = 0$ we have $z = a$ and hence since $r^2 = 4az$ initially we have $r = 2a$. Moreover, since the particle is projected *horizontally* at speed $v$, in polar coordinates (see Figure 13) we may identify $\dot{r} = 0$, $r\dot{\theta} = v$ initially. We thus compute

$$h = r^2\dot{\theta} = r \cdot r\dot{\theta} = 2av . \tag{5.35}$$

Conservation of energy (5.32) reads

$$E = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + mgz = \frac{1}{2}mv^2 + mga , \tag{5.36}$$

where we have substituted the initial conditions into the second equality. Thus

$$\frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{z}^2\right) + gz = \frac{1}{2}v^2 + ga . \tag{5.37}$$

Let us now eliminate $\dot{\theta}$ and $r$ to get a differential equation for $z(t)$ only. From the constraint

$$r = 2\sqrt{az} \qquad \Longrightarrow \qquad \dot{r} = \sqrt{\frac{a}{z}}\,\dot{z}\ . \tag{5.38}$$

Substituting into (5.37) using $\dot{\theta} = h/r^2$ gives

$$\frac{1}{2}\left[\left(1+\frac{a}{z}\right)\dot{z}^2 + \frac{4a^2v^2}{4az}\right] + gz = \frac{1}{2}v^2 + ga\ . \tag{5.39}$$

This may be expanded out and simplified to

$$\left(1+\frac{a}{z}\right)\dot{z}^2 = v^2\left(1-\frac{a}{z}\right) + 2g(a-z) = \frac{1}{z}\left(v^2-2gz\right)(z-a)\ . \tag{5.40}$$

Since $z > 0$ and $\dot{z}^2 \geq 0$ it follows that

$$\left(z - \frac{v^2}{2g}\right)(z-a) \leq 0\ . \tag{5.41}$$

Therefore the particle always stays between the two heights $z = a$ and $z = v^2/2g$, at which $\dot{z} = 0$. In particular the particle is confined to $z \geq a$ if $v^2 > 2ga$, or to $z \leq a$ if $v^2 < 2ga$, or to the horizontal circle $z = a$ if $v^2 = 2ga$. ■

# 6 The Kepler problem

In this section we introduce Newton's *law of universal gravitation*. This is described by an *inverse square law force*, and we show that a particle acted on by such a force moves on a conic section. This was famously first shown by Newton in his *Principia*. We also derive Kepler's laws of planetary motion, and comment briefly on the inverse square law force of electrostatics.

## 6.1 Inverse square law forces and potentials

In sections 4.2 and 4.3 we introduced the notions of *conservative forces* and *central forces*. These lead to a conserved energy and conserved angular momentum, respectively. In this section we combine the two. Specifically, we are interested in forces given by the following:

**Proposition** Denote $r = |\mathbf{r}|$ and $\mathbf{e}_r = \mathbf{r}/r = \hat{\mathbf{r}}$ a unit vector in the direction of $\mathbf{r}$, where the latter is the position vector of a particle. Then forces of the form

$$\mathbf{F} = F(r)\,\mathbf{e}_r \ , \tag{6.1}$$

are *conservative central forces*, where the potential $V = V(r)$ depends only on the distance $r$ to the origin.

**Proof**: It is immediate that (6.1) is a central force, as it is proportional to $\mathbf{r}$. Suppose that $\mathbf{F}$ is conservative, of the form $\mathbf{F} = -\nabla V(r)$. We introduce Cartesian coordinates $\mathbf{r} = (r_1, r_2, r_3)$, so that $r = \sqrt{r_1^2 + r_2^2 + r_3^2}$. From this it follows that $\partial r/\partial r_i = r_i/r$, $i = 1, 2, 3$. Using the chain rule we hence compute

$$\mathbf{F} = -\nabla V = -\left(\frac{\partial V}{\partial r_1}, \frac{\partial V}{\partial r_2}, \frac{\partial V}{\partial r_3}\right) = -\frac{\mathrm{d}V}{\mathrm{d}r}\left(\frac{r_1}{r}, \frac{r_2}{r}, \frac{r_3}{r}\right) = -\frac{\mathrm{d}V}{\mathrm{d}r}\,\mathbf{e}_r \ . \tag{6.2}$$

Thus $F(r)$ in (6.1) is

$$F(r) = -\frac{\mathrm{d}V}{\mathrm{d}r} \ . \tag{6.3}$$

Conversely, given a central force of the form (6.1) we may simply define the potential via $V(r) = -\int_{r_0}^{r} F(s)\,\mathrm{d}s$. ■

More specifically, for the remainder of this section we are interested in the following important example:

**Definition** The *inverse square law force* is a conservative central force with

$$V(r) = -\frac{\kappa}{r} \qquad \Longrightarrow \qquad \mathbf{F} = -\frac{\kappa}{r^2}\,\mathbf{e}_r \ , \tag{6.4}$$

where $\kappa$ is constant, and we have used (6.1) and (6.3) to relate the potential to the force.

Inverse square law forces arise in Nature in two different contexts:

**Newton's law of universal gravitation**

According to Newton, the gravitational force on a point particle at position $\mathbf{r}_1$ due to a point particle at position $\mathbf{r}_2$ is given by (see Figure 17)

$$\mathbf{F} = \mathbf{F}_{12} = -G_N \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \frac{(\mathbf{r}_1 - \mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|} = -G_N \frac{m_1 m_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2} \hat{\mathbf{r}}_{12} . \tag{6.5}$$

Here $m_1$, $m_2$ are the (gravitational) masses of the two particles, we have defined the unit vector $\hat{\mathbf{r}}_{12} = (\mathbf{r}_1 - \mathbf{r}_2)/|\mathbf{r}_1 - \mathbf{r}_2|$, and $G_N \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$ is *Newton's gravitational constant.* Note that:

- The force is proportional to the product of the masses; given the overall minus sign and the fact that masses are positive, the gravitational force is always *attractive.*
- The force acts in the direction of the vector joining the two masses, and is inversely proportional to the square of the distance of separation.

In fact Newton's law of universal gravitation (6.5) is equivalent to these two statements. ■

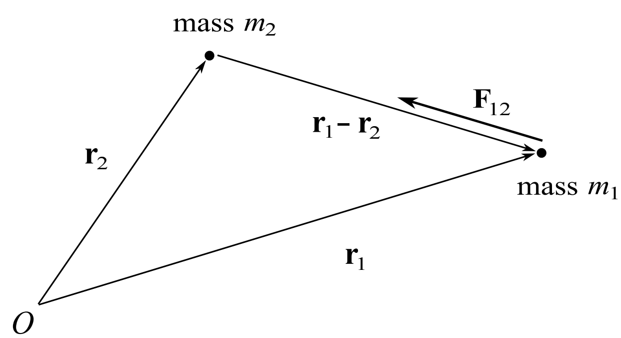



Figure 17: The attractive gravitational force $\mathbf{F}_{12}$ on a mass $m_1$ at position $\mathbf{r}_1$, due to a mass $m_2$ at position $\mathbf{r}_2$. A particular physical case of interest has the Earth for mass $m_1$ and the Sun for mass $m_2$.

**Remark**: We now apparently have *two* different descriptions of the force of gravity: one given by Newton's inverse square law force, and the other given by $\mathbf{F} = -mg\,\mathbf{k}$. We shall derive the latter from the former below.

Let us now put the second mass at the *origin* $O$ (that is, we put $\mathbf{r}_2 = \mathbf{0}$), and relabel $m_2 = M$. We also write $\mathbf{r}_1 = \mathbf{r}$ and $m_1 = m$. Then we may restate Newton's law of universal gravitation (6.5) in this set up as:

> A point mass $M$ at the origin $O$ exerts a gravitational force $\mathbf{F}$ on a point mass $m$ at position $\mathbf{r}$ given by (6.4), where $\kappa = G_N M m > 0$.

**Remark**: By Newton's third law **N3** there will be an equal and opposite force on the mass at the origin, so unless something else is holding it there it will accelerate and hence not remain at the origin. We shall return to precisely this point in section 7.3.

Consider the particle of mass $M$ sitting at the origin $O$. If we now place an additional particle of mass $m$ at position $\mathbf{r}$, this particle has potential energy $V(r) = -G_N Mm/r$ due to the attractive force it experiences towards the mass $M$. This leads to the following definition:

**Definition** The *Newtonian gravitational potential*, or *Newtonian gravitational field*, generated by the mass $M$ is

$$\Phi(\mathbf{r}) \quad = \quad -\frac{G_N M}{r} \ . \tag{6.6}$$

The potential energy of the mass $m$ is then simply $V = m\Phi$, giving total conserved energy

$$E \quad = \quad \frac{1}{2}m|\dot{\mathbf{r}}|^2 + m\Phi(\mathbf{r}) \ . \tag{6.7}$$

Recalling that forces are additive, it follows that the Newtonian gravitational potential generated by point masses $M_1, \ldots, M_N$ at positions $\mathbf{r}_1, \ldots, \mathbf{r}_N$ is

$$\Phi(\mathbf{r}) \quad = \quad -G_N \sum_{I=1}^{N} \frac{M_I}{|\mathbf{r} - \mathbf{r}_I|} \ . \tag{6.8}$$

To see this, note that the total force acting on a mass $m$ at position $\mathbf{r}$ is correctly given by

$$\mathbf{F} \quad = \quad -\nabla(m\Phi) \quad = \quad -G_N m \sum_{I=1}^{N} \frac{M_I}{|\mathbf{r} - \mathbf{r}_I|^3}(\mathbf{r} - \mathbf{r}_I) \ . \tag{6.9}$$

The potential energy of the mass $m$ is again $V = m\Phi$, with conserved energy (6.7).

We'd like to now *derive* the formula $\mathbf{F} = -mg\,\mathbf{k}$, where $g$ is acceleration due to gravity at the Earth's surface. This is called a *uniform gravitational field*, to distinguish it from the more general law of universal gravitation in which $\mathbf{F}$ is not (approximately) constant. We first need the following result, also due to Newton. The proof of this is starred for this course, but it is part of the Multivariable Calculus course:

**Proposition** (Newton's Shell Theorem) The Newtonian gravitational potential *external* to a *spherically symmetric* body of total mass $M$ is the same as that generated by a point mass $M$ at the centre of mass. That is, the gravitational potential is given by (6.6), where the origin is at the centre of mass.

* **Proof**: We treat the body as a continuous distribution of mass, with the mass contained in a small volume $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ centred at the point $\mathbf{x}$ being $\rho(\mathbf{x})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. The function $\rho(\mathbf{x})$ is called the *density*. The continuum limit of (6.8) then reads

$$\Phi(\mathbf{r}) \quad = \quad -G_N \iiint \frac{\rho(\mathbf{x})}{|\mathbf{r} - \mathbf{x}|}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \ , \tag{6.10}$$

where the integral is over points $\mathbf{x}$ in the body. Being *spherically symmetric* means that $\rho$ depends only on the distance from the centre of mass of the body, which we take to be at the origin $O$. In spherical polar coordinates $(R, \theta, \varphi)$ for the point $\mathbf{x}$ then $\rho = \rho(R)$. Without loss of generality we take $\mathbf{r}$ to lie along the polar axis $\theta = 0$, so that $\mathbf{r} \cdot \mathbf{x} = rR\cos\theta$, and hence $|\mathbf{r} - \mathbf{x}| = \sqrt{r^2 + R^2 - 2rR\cos\theta}$. We also have $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = R^2 \sin\theta\,\mathrm{d}R\,\mathrm{d}\theta\,\mathrm{d}\varphi$. The integral (6.10) thus reads

$$\Phi(\mathbf{r}) \quad = \quad -G_N \int_{R=0}^{R_0} \mathrm{d}R \int_{\theta=0}^{\pi} \mathrm{d}\theta \int_{\varphi=0}^{2\pi} \mathrm{d}\varphi \, \frac{\rho(R)\, R^2 \sin\theta}{\sqrt{r^2 + R^2 - 2rR\cos\theta}} \; . \tag{6.11}$$

Here $R_0$ is the radius of the body. Nothing in the integral depends on $\varphi$, so the integral over the latter just gives an overall factor of $2\pi$. We can also do the $\theta$ integral, which gives

$$\begin{aligned} \Phi(\mathbf{r}) \quad &= \quad -2\pi G_N \int_{R=0}^{R_0} \mathrm{d}R\, \rho(R)\, R^2 \cdot \frac{1}{rR} \left[\sqrt{r^2 + R^2 - 2rR\cos\theta}\right]_{\theta=0}^{\theta=\pi} \\ &= \quad -\frac{2\pi G_N}{r} \int_0^{R_0} \mathrm{d}R\, \rho(R)\, R\left(|r+R| - |r-R|\right) \; . \end{aligned} \tag{6.12}$$

For $\mathbf{r}$ external to the body we have $r > R_0 \geq R$, and thus $|r+R| - |r-R| = 2R$, giving

$$\Phi(\mathbf{r}) \quad = \quad -\frac{4\pi G_N}{r} \int_0^{R_0} \mathrm{d}R\, \rho(R)\, R^2 \quad = \quad -\frac{G_N M}{r} \; . \tag{6.13}$$

This is precisely what we wanted to prove. Here in the last step we have used the fact that the total mass is by definition

$$M \quad \equiv \quad \iiint \rho(\mathbf{x})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \quad = \quad 4\pi \int_0^{R_0} \mathrm{d}R\, \rho(R)\, R^2 \; , \tag{6.14}$$

where the factor of $4\pi$ comes from integrating over $\theta$ and $\varphi$. ■

Consider now a particle of mass $m$ near the Earth's surface. From the above Proposition we may assume that all the mass of the Earth is concentrated at its centre. We take $M = M_E \simeq 5.97 \times 10^{24}$ kg to be the Earth's mass. Our particle has position vector $\mathbf{r} = (R_E + z)\,\mathbf{k}$, where $R_E \simeq 6.37 \times 10^6$ m is the radius of the Earth and $\mathbf{k}$ is a unit vector pointing radially outwards from the centre of the Earth. The law of universal gravitation then gives the force on the mass $m$ as

$$\mathbf{F} \quad = \quad -G_N \frac{mM_E}{(R_E + z)^2}\,\mathbf{k} \quad \simeq \quad -\frac{G_N M_E}{R_E^2}\, m\,\mathbf{k} \; . \tag{6.15}$$

We hence identify $g = G_N M_E / R_E^2 \simeq 9.81\ \mathrm{m\,s^{-2}}$. The approximation in (6.15) holds for distances $z$ small compared to the radius of the Earth.

Here is another example of the use of this result:

**Example** (Escape velocity): From (6.7) and the last Proposition the conserved energy of a particle of mass $m$ in the gravitational field of the Earth is

$$E \quad = \quad \frac{1}{2} m v^2 - \frac{G_N M_E\, m}{r} \; , \tag{6.16}$$

where $v = |\dot{\mathbf{r}}|$ is the speed of the particle, and $r$ is its distance from the centre of (mass of) the Earth. Suppose the particle starts with an initial speed $v_*$ at distance $r_*$, and escapes to $r = \infty$ with speed $v_\infty$. By conservation of energy

$$\frac{1}{2}mv_*^2 - \frac{G_N M_E\, m}{r_*} \quad = \quad \frac{1}{2}mv_\infty^2 \;\geq\; 0 \; . \tag{6.17}$$

It follows that $v_* \geq v_e$, where the *escape velocity* $v_e$ from the radius $r_*$ is

$$v_e \quad = \quad \sqrt{\frac{2G_N M_E}{r_*}} \; . \tag{6.18}$$

From the surface of the earth $r_* \simeq 6.37 \times 10^6$ m, and using $M_E \simeq 5.97 \times 10^{24}$ kg, $G_N \simeq 6.67 \times 10^{-11}\ \mathrm{N\,m^2\,kg^{-2}}$ we compute the escape velocity $v_e \simeq 11.2 \times 10^3\ \mathrm{m\,s^{-1}}$ – about 25,000 miles per hour! Notice this ignores air resistance, which will only increase this speed. ■

**Coulomb's law of electrostatics**

Coulomb discovered a similar inverse square law force between two point charges at rest. Given two such charges $q_1$, $q_2$ at positions $\mathbf{r}_1$, $\mathbf{r}_2$, respectively, the first charge experiences an *electrostatic* force $\mathbf{F}_{12}$ due to the second charge, given by

$$\mathbf{F}_{12} \quad = \quad \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\hat{\mathbf{r}}_{12} \; . \tag{6.19}$$

The constant $\epsilon_0 \simeq 8.85 \times 10^{-12}\ \mathrm{C^2\,N^{-1}\,m^{-2}}$ is called the *permittivity of free space.* Unlike gravity, the Coulomb force can be both attractive and repulsive, with opposite sign charges attracting, and same sign charges repelling.

As we did for gravity, let us now put the second charge at the origin ($\mathbf{r}_2 = \mathbf{0}$), and relabel $q_2 = Q$. We also write $\mathbf{r}_1 = \mathbf{r}$ and $q_1 = q$. Then we may restate Coulomb's law as:

> A point charge $Q$ at the origin $O$ exerts an electrostatic force $\mathbf{F}$ on a point charge $q$ at position $\mathbf{r}$ given by (6.4), where $\kappa = -Qq/4\pi\epsilon_0$.

In *Electrostatics* (the study of charges at rest), the *electric field* $\mathbf{E} = \mathbf{E}(\mathbf{r})$ is by definition the force a unit test charge (*i.e.* $q = 1$) at rest experiences at the position $\mathbf{r}$. For example, Coulomb's law (6.19) implies that a point charge $Q$ at the origin generates an electric field

$$\mathbf{E}(\mathbf{r}) \quad = \quad \frac{1}{4\pi\epsilon_0}\frac{Q}{|\mathbf{r}|^2}\,\hat{\mathbf{r}} \; . \tag{6.20}$$

The force on another charge $q$ at position $\mathbf{r}$ is then by definition $\mathbf{F} = q\,\mathbf{E}(\mathbf{r})$. In electromagnetism it turns out there is an *electric potential* $\phi = \phi(\mathbf{r})$ for which

$$\mathbf{E} \quad = \quad -\nabla\phi \; . \tag{6.21}$$

For example, from (6.20) the electric potential generated by a point charge $Q$ at the origin is

$$\phi(\mathbf{r}) \quad = \quad \frac{Q}{4\pi\epsilon_0 r} \; . \tag{6.22}$$

The potential energy of a charge $q$ at position $\mathbf{r}$ is then simply $V = q\phi$, and the total energy

$$E \;=\; \frac{1}{2}m|\dot{\mathbf{r}}|^2 + q\phi(\mathbf{r}) \tag{6.23}$$

is conserved (see the Proposition below). This is all starting to look very similar to Newtonian gravity! However, if you continue to study the theories of gravity and electromagnetism further in Parts B and C you'll see that at a deeper level the two theories are very different. We conclude by proving the following:

**Proposition** The energy (6.23) is conserved.

**Proof**: We compute

$$\dot{E} \;=\; m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} + q\nabla\phi\cdot\dot{\mathbf{r}} \;=\; m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} - q\,\mathbf{E}\cdot\dot{\mathbf{r}}\ . \tag{6.24}$$

Recall that in general the *total* force acting on the particle is given by the Lorentz force law (2.26): $\mathbf{F} = q\,\mathbf{E} + q\,\dot{\mathbf{r}}\wedge\mathbf{B}$. Thus

$$\dot{E} \;=\; m\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}} + (q\,\dot{\mathbf{r}}\wedge\mathbf{B} - \mathbf{F})\cdot\dot{\mathbf{r}} \;=\; (m\ddot{\mathbf{r}} - \mathbf{F})\cdot\dot{\mathbf{r}} \;=\; 0\ . \tag{6.25}$$

Here the last equality is Newton's second law, and the second equality follows since $(\dot{\mathbf{r}}\wedge\mathbf{B})\cdot\dot{\mathbf{r}} = 0$. The latter means that the magnetic component $\mathbf{F}_{\mathrm{mag}} = q\,\dot{\mathbf{r}}\wedge\mathbf{B}$ of the Lorentz force law does no work. Thus although $\mathbf{F}_{\mathrm{mag}}$ depends on velocity, we nevertheless have conservation of energy. ■

* In general the potential $\phi = \phi(\mathbf{r}, t)$ also depends on time $t$, and the above discussion is modified. But we'll leave this for the course B7.2.

## 6.2 The Kepler problem and planetary orbits

Let us return to the conservative central force (6.1),

$$\mathbf{F} \;=\; F(r)\,\mathbf{e}_r\ . \tag{6.26}$$

We assume this force acts on a particle of mass $m$, and is directed towards a *fixed* centre $O$, which is the origin of an inertial frame. From section 4.3 we know that angular momentum $\mathbf{L} = \mathbf{L}_O$ about the origin is conserved, and that the motion of the particle lies in a plane. We introduce the polar coordinates of section 4.1 in this plane. Using the formula (4.6) for acceleration $\ddot{\mathbf{r}}$ in polar coordinates, Newton's second law

$$m\ddot{\mathbf{r}} \;=\; F(r)\,\mathbf{e}_r \tag{6.27}$$

becomes

$$m\left[(\ddot{r} - r\dot{\theta}^2)\,\mathbf{e}_r + \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot{\theta})\,\mathbf{e}_\theta\right] \;=\; F(r)\,\mathbf{e}_r\ . \tag{6.28}$$

We then read off two scalar equations from (6.28):

$$\frac{\mathrm{d}}{\mathrm{d}t}(r^2\dot\theta) = 0 \,, \tag{6.29}$$

$$m(\ddot r - r\dot\theta^2) = F(r) \,. \tag{6.30}$$

We recognize the first equation (6.29) as a consequence of angular momentum conservation, as discussed in section 4.3. Indeed, recall that $\mathbf{L} = mr^2\dot\theta\,\mathbf{k} = mh\,\mathbf{k}$, where as earlier $\mathbf{k} = \mathbf{e}_r \wedge \mathbf{e}_\theta$ is orthogonal to the plane of motion, and we define

$$r^2\dot\theta \equiv h = \text{constant} \,, \tag{6.31}$$

where $h$ is the specific angular momentum. Using (6.31) to substitute for $\dot\theta$ in terms of $h$ into (6.30) gives

$$m\left(\ddot r - \frac{h^2}{r^3}\right) = F(r) \,. \tag{6.32}$$

Using conservation of angular momentum, we have reduced motion in a conservative central force to a second order ODE for the distance of the particle from the origin! Solving this gives $r(t)$, which we may then substitute into (6.31) to obtain a first order ODE for $\theta(t)$, namely $\dot\theta = h/r(t)^2$. Eliminating time $t$ from this solution will generically give some curve $r = r(\theta)$ (compare with the projectile example in section 2.1, where we first solved for the trajectory as a function of time $t$, and from this then eliminated $t$ to find the curve). For our particular problem it turns out to be easier to solve for $r(\theta)$ directly, or rather $u(\theta) \equiv 1/r(\theta)$.

**Proposition** For a particle moving in a central force the equations of motion (6.31), (6.32) imply that, for $h \neq 0$,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = -\frac{F(1/u)}{mh^2u^2} \,, \tag{6.33}$$

where $u(\theta) = 1/r(\theta)$ gives the curve traced out by the path of the particle. Having solved (6.33) we may restore the time-dependence by solving $\dot\theta = h\,u(\theta)^2$ to find $\theta(t)$.

**Proof**: From (6.31) we have $\dot\theta = hu^2$, giving

$$\dot r = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{\mathrm{d}u}{\mathrm{d}\theta}\,\dot\theta = -h\frac{\mathrm{d}u}{\mathrm{d}\theta} \,. \tag{6.34}$$

Differentiating again:

$$\ddot r = \frac{\mathrm{d}}{\mathrm{d}t}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) = -h\dot\theta\,\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = -h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} \,. \tag{6.35}$$

Substituting this into (6.32) gives

$$m\left(-h^2u^2\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} - h^2u^3\right) = F\left(\frac{1}{u}\right) \,, \tag{6.36}$$

which rearranges to (6.33). ■

Notice that (6.33) is not valid in the special case that $h = 0$. Since $|\mathbf{L}| = m|h|$ in fact $h = 0$ if and only if the angular momentum $\mathbf{L} = \mathbf{0}$, and from our discussion in section 4.3 we know this means the particle must be travelling on a straight line through the origin – this can be seen explicitly from (6.31), which implies that $\dot{\theta} = 0$. In this case $\theta$ is constant, and it doesn't make sense to parametrize $r = r(\theta)$ as we do in the Proposition. Solutions with $h = 0$ are called *radial trajectories*, and are studied in Problem Sheet 5. For the remainder of this section we assume that $\mathbf{L} \neq \mathbf{0}$.

We now examine the central inverse square law force, with $F(r) = -\kappa/r^2$. This is called the *Kepler problem*. Recall that $\kappa > 0$ for an attractive force (such as gravity), while $\kappa < 0$ for a repulsive force (such as the electrostatic force between two charges with the same sign). Until section 6.4 we'll focus on the attractive case, with $\kappa = G_N Mm > 0$. The following theorem is the main result of this section.

**Theorem** (Due to Newton) For the Kepler problem the particle trajectories with non-zero angular momentum are conic sections.

**Proof**: In terms of the variable $u = 1/r$ we have $F(r) = -\kappa u^2$. Substituting this into (6.33) gives

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u \quad = \quad \frac{\kappa}{mh^2} \ . \tag{6.37}$$

Remarkably, the change of variable has reduced the problem to the same ODE we found for a particle attached to a spring (*c.f.* equation (2.21))! The general solution to (6.37) is

$$u(\theta) \quad = \quad \frac{\kappa}{mh^2}\left[1 + e\,\cos\left(\theta + \phi\right)\right] \ , \tag{6.38}$$

where $e$ and $\phi$ are integration constants. Without loss of generality we may assume that $e \geq 0$, and then further using the freedom to rotate the plane we may assume that $\phi = 0$, which we henceforth do. On the other hand, from the Prelims Geometry course we know that the general polar form of a conic may be written as

$$\frac{r_0}{r} \quad = \quad r_0 u \quad = \quad 1 + e\,\cos\theta \ , \tag{6.39}$$

where $r_0$ is a constant and the origin at $r = 0$ is situated at one of the foci. Comparing to (6.38) and recalling that $\kappa > 0$ we may thus identify

$$r(\theta) \quad = \quad \frac{r_0}{1 + e\cos\theta} \ , \qquad \text{where} \qquad r_0 \quad = \quad \frac{mh^2}{\kappa} \ > \ 0 \ . \tag{6.40}$$

Regarding $m$ and $\kappa$ as fixed, the scale parameter $r_0$ is thus determined by the specific angular momentum $h$. The integration constant $e$ in (6.38) is the eccentricity of the conic. This is (i) an ellipse for $0 \leq e < 1$, with $e = 0$ being a circle, (ii) a parabola for $e = 1$, and (iii) a hyperbola for $e > 1$. ■

Notice that the time dependence is recovered by solving $\dot{\theta} = h\,u(\theta)^2$ as

$$h\,t \;=\; \int \frac{\mathrm{d}\theta}{u(\theta)^2} \;=\; r_0^2 \int \frac{\mathrm{d}\theta}{(1+e\,\cos\theta)^2}\;, \tag{6.41}$$

which gives $t$ as a function of $\theta$. (It's possible to do the integral on the right hand side, but we won't pursue this further.)

This Theorem is such an iconic result in Dynamics that we'll spend the rest of this section analysing various aspects of the problem and solution in more detail. We start with a brief reminder of the geometry of conics – this should be revision from the Geometry course.

**Conics**

We begin by expressing the polar form of a conic (6.40) in Cartesian coordinates $x = r\cos\theta$, $y = r\sin\theta$. We first rearrange (6.40) to give

$$r_0 \;=\; e\,r\cos\theta + r \;=\; e\,x + r \qquad \Longrightarrow \qquad r \;=\; r_0 - e\,x\;. \tag{6.42}$$

Squaring both sides then gives

$$x^2 + y^2 \;=\; (r_0 - e\,x)^2\;. \tag{6.43}$$

How we proceed now depends on $e$.

**Ellipses:** $0 \leq e < 1$: In this case we define

$$a^2 \;=\; \frac{r_0^2}{(1-e^2)^2}\;, \qquad b^2 \;=\; \frac{r_0^2}{1-e^2}\;, \qquad x_0 \;=\; -\frac{e\,r_0}{1-e^2} \;=\; -e\,a\;, \tag{6.44}$$

taking the positive square roots for $a$ and $b$. Note that $x_0 \leq 0$. After a little algebra (6.43) becomes[15]

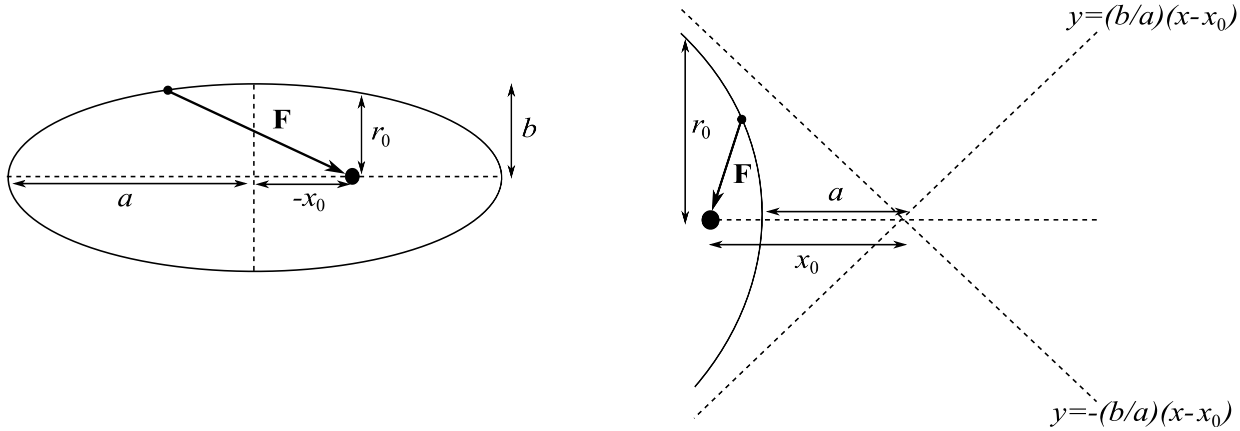
$$\frac{(x-x_0)^2}{a^2} + \frac{y^2}{b^2} \;=\; 1\;. \tag{6.45}$$

This is the equation of an ellipse with centre $(x_0, 0)$ and semi-major axis of length $a$ and semi-minor axis of length $b \leq a$. One of the two foci is at the origin $(x, y) = (0, 0)$, which is the centre of attraction $r = 0$ for the inverse square law force (when $\kappa > 0$). See Figure 18a. The ellipse is a circle for $e = 0$, when the centre of the ellipse is at the origin and $a = b = r_0$.

**Hyperbolae:** $e > 1$: In this case we similarly define

$$a^2 \;=\; \frac{r_0^2}{(e^2-1)^2}\;, \qquad b^2 \;=\; \frac{r_0^2}{e^2-1}\;, \qquad x_0 \;=\; \frac{e\,r_0}{e^2-1} \;=\; e\,a\;, \tag{6.46}$$

[15] If you want to check this it's easiest to start with the left hand side of (6.45) and show this equals 1 using (6.44) and (6.43).




(a) An ellipse. The large black dot is the origin, which is one of the foci and also the centre of attraction (for $\kappa > 0$) of the inverse square law force. The centre of the ellipse is $(x_0, 0)$, where $x_0 = -e\,a \leq 0$. The semi-major axis has length $a$, while the semi-minor axis has length $b \leq a$.

(b) A hyperbola. The large black dot is again the origin, focus, and centre of the force. The two asymptotes are $y = \pm(b/a)(x-x_0)$, which meet at the point $(x_0, 0)$, where now $x_0 = e\,a > 0$.

Figure 18: Conic sections.

again taking positive square roots for $a$ and $b$. Notice that now $x_0 > 0$. Some algebra reveals that (6.43) becomes

$$\frac{(x-x_0)^2}{a^2} - \frac{y^2}{b^2} \quad = \quad 1 \ . \tag{6.47}$$

This is the equation of a hyperbola. There are two asymptotes $y = \pm(b/a)(x - x_0)$, dropping the "1" from the right hand side of (6.47), which meet at $x = x_0$. See Figure 18b. The focus is at the origin $(x, y) = (0, 0)$, which is again the centre of the inverse square law force. Notice from (6.40) that $r \to \infty$ along the asymptotes for $\cos\theta = -1/e$, which has two solutions $\theta = \pm\theta_0$, where $\theta_0 = \cos^{-1}(-1/e) > \pi/2$ and $\theta$ is the angle subtended at the origin (large black dot in Figure 18b).

**Parabolae:** $e = 1$: Equation (6.43) reads simply

$$y^2 \quad = \quad r_0^2 - 2r_0 x \ , \tag{6.48}$$

which is the equation of a parabola. This is again an unbounded orbit, where now $r \to \infty$ for $\cos\theta = -1$, *i.e.* $\theta = \pm\pi$.

### The effective potential and energy

Let's return to the original equation of motion (6.32) for $r(t)$. Recalling that $F(r) = -\mathrm{d}V/\mathrm{d}r$ we may write (6.32) as

$$m\ddot{r} \quad = \quad -\frac{\mathrm{d}V_{\mathrm{eff}}}{\mathrm{d}r} \ , \tag{6.49}$$

where we have introduced the *effective potential* (careful with signs!)

$$V_{\mathrm{eff}}(r) \quad = \quad V(r) + \frac{mh^2}{2r^2} \ . \tag{6.50}$$

The equation of motion (6.49) now resembles motion in one dimension, with an effective potential energy $V_{\rm eff}$. Indeed, we know that the energy of the particle

$$E = \frac{1}{2}m|\dot{\mathbf{r}}|^2 + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \qquad (6.51)$$

is conserved. Substituting for $\dot{\theta}$ in terms of $h$ using (6.31) gives

$$E = \frac{1}{2}m\dot{r}^2 + V_{\rm eff}(r) \ . \qquad (6.52)$$

The equation of motion (6.49) indeed implies this is conserved, as we learned in section 3.1.

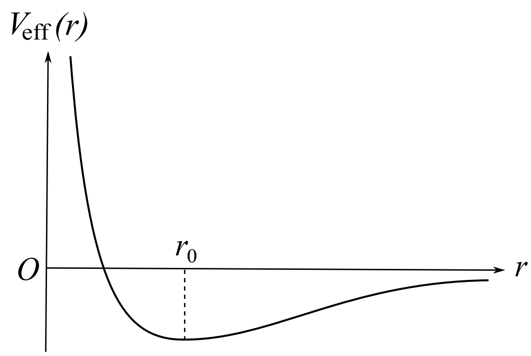



Figure 19: The effective potential $V_{\rm eff}(r)$ for the Kepler inverse square law force problem, crucially with $\kappa > 0$ so that the force is attractive. In this case $V_{\rm eff}$ has a unique local minimum at $r = r_0$.

For the Kepler problem we have $V(r) = -\kappa/r$, with an attractive force such as gravity having $\kappa > 0$. The effective potential is shown in Figure 19. A solution with $r = r_0$ constant has $\ddot{r} = 0$, and thus from (6.49) $r_0$ must be a critical point of the effective potential. One easily checks that

$$\frac{{\rm d}V_{\rm eff}}{{\rm d}r}(r_0) = 0 \qquad \Longrightarrow \qquad r_0 = \frac{mh^2}{\kappa} \ . \qquad (6.53)$$

Thus such a solution exists if and only if $\kappa > 0$. Of course an orbit with $r = r_0$ constant is a circle, and this is consistent with our general solution (6.40) with eccentricity $e = 0$. Being a local minimum of the effective potential also means that this circular orbit is *stable* to small perturbations of $r$, as we learned in section 3.3.

**Example** (Geostationary orbit): A *geostationary orbit* is a circular orbit in the plane containing the Earth's equator, which co-rotates with the Earth. This means that a satellite following such a trajectory lies directly above the same point on the Earth's surface, maintaining the same height. It hence has the same angular velocity as the Earth about its polar axis, namely $\dot{\theta} = 2\pi$ radians per day. Using equation (6.31) we may write $h = r_0^2\,\dot{\theta}$, and since $\kappa = G_N M_E\, m$, where $M_E$ is the mass of the Earth, (6.53) implies the radius satisfies

$$r_0 = \frac{mh^2}{\kappa} = \frac{r_0^4\,\dot{\theta}^2}{G_N M_E} \quad \Longrightarrow \quad r_0 = \left(\frac{G_N M_E}{\dot{\theta}^2}\right)^{1/3} \simeq 4.22 \times 10^7 \text{ m} = 42,200 \text{ km} \ . \quad (6.54)$$

Here we've used $G_N \simeq 6.67 \times 10^{-11}\,\mathrm{N\,m^2\,kg^{-2}}$, $M_E \simeq 5.97 \times 10^{24}$ kg, $\dot{\theta} \simeq 7.27 \times 10^{-5}\ \mathrm{s}^{-1}$. ■

Next we'd like to compute the conserved energy $E$ in (6.52). For this we need $\dot{r}$, which from (6.34) is

$$\dot{r} \;=\; -h\,\frac{\mathrm{d}u}{\mathrm{d}\theta} \;=\; \frac{he}{r_0}\sin\theta\ , \tag{6.55}$$

where in the last equality we have used the form of the solution in (6.39). Inserting this and $r = r_0/(1 + e\,\cos\theta)$ into the energy

$$E \;=\; \frac{1}{2}m\dot{r}^2 - \frac{\kappa}{r} + \frac{mh^2}{2r^2}\ , \tag{6.56}$$

and substituting for $r_0$ using (6.53) we find that $E$ is indeed constant:

$$E \;=\; \frac{(e^2-1)\kappa^2}{2mh^2}\ . \tag{6.57}$$

In particular we see that the bound orbits with $0 \le e < 1$ (*i.e.* ellipses) have $E < 0$. But this is also clear from the effective potential in Figure 19: for $E < 0$ the particle moves back and forth between some $r_{\mathrm{min}}$ and $r_{\mathrm{max}}$, and the orbit is bound, *c.f.* our discussion of motion in a general potential in section 3.2. On the other hand for $e > 1$ we have $E > 0$ and the particle has a minimum radius, but escapes to infinity. These are the hyperbolic orbits. The parabola $e = 1$ is the limiting case with zero energy, for which the particle only just escapes to infinity.

We conclude this subsection with an astrophysical example:

**Example** (Angle of deflection of a comet): A comet approaches the Sun from a very large distance with speed $v$. If the Sun exerted no force on the comet it would continue with uniform velocity on an undeflected path, giving a distance of closest approach to the Sun of $p$. Find the actual path of the comet and the approximate angle through which it is deflected.

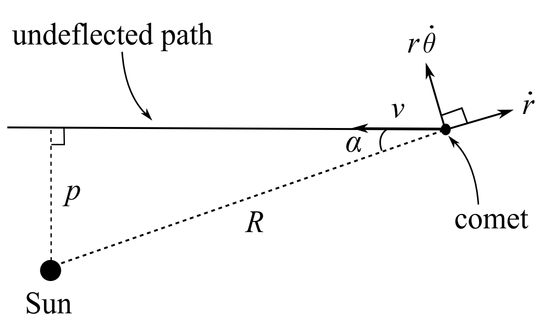



Figure 20: A comet approaching the Sun from a very large distance $R$ with speed $v$. Without the effect of gravity the comet travels undeflected with constant speed $v$, and its closest approach is the distance $p$. Here $p = R\sin\alpha$, and the angle $\alpha$ is very small.

**Solution**: Figure 20 shows the comet's path undeflected by gravity. At time $t = -T$, for some large $T \gg 0$, we have $\dot{r} = -v \cos\alpha$ and $r\dot{\theta} = v \sin\alpha = p\, v/R$, where in the latter equation we have used $p = R \sin\alpha$. In particular the conserved specific angular momentum $h$ may be computed from these initial conditions as

$$h \;=\; r^2\dot{\theta} \;=\; p\, v \; . \tag{6.58}$$

The general solution to the Kepler problem may be written

$$u(\theta) \;=\; \frac{\kappa}{mh^2} + C\, \cos\theta + D\, \sin\theta \; . \tag{6.59}$$

For this example it turns out to be more convenient to use this form, rather than (6.38). At time $t = -T$ we choose the axes so that $\theta = 0$. In addition we have $u = 1/R \simeq 0$ and from (6.34) also $\mathrm{d}u/\mathrm{d}\theta = -\dot{r}/h = \cos\alpha/p \simeq 1/p$. Inserting these initial conditions into (6.59) determines the integration constants $C$ and $D$, giving solution

$$u(\theta) \;=\; \frac{\kappa}{mp^2v^2}(1 - \cos\theta) + \frac{1}{p}\sin\theta \; . \tag{6.60}$$

This is the path of the comet.

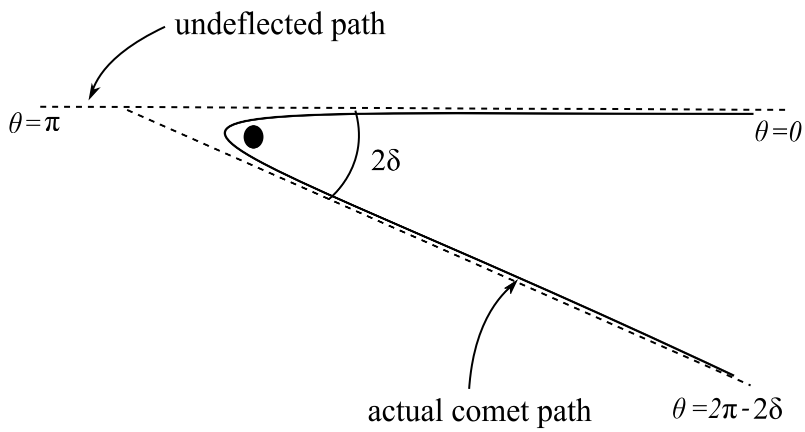



Figure 21: The actual path of the comet. The origin is at the Sun (large black dot), with the $\theta = 0$ axis horizontal, to the right (one should understand the dotted lines as extending to infinity).

Setting $u = 1/r = 0$ gives the equation

$$\frac{\kappa}{mp^2v^2}(1 - \cos\theta) + \frac{1}{p}\sin\theta \;=\; 0 \; . \tag{6.61}$$

Using double angle formulas we may rewrite this as

$$\frac{\kappa}{mp^2v^2}\sin^2\frac{\theta}{2} + \frac{1}{p}\sin\frac{\theta}{2}\cos\frac{\theta}{2} \;=\; 0 \; . \tag{6.62}$$

Clearly one solution is $\theta = 0$, but there is another solution satisfying

$$\frac{\kappa}{mp^2v^2}\sin\frac{\theta}{2} + \frac{1}{p}\cos\frac{\theta}{2} \;=\; 0 \qquad \Longrightarrow \qquad \tan\frac{\theta}{2} \;=\; -\frac{mpv^2}{\kappa} \; . \tag{6.63}$$

Setting $\theta = 2\pi - 2\delta$ this may be rewritten as

$$\tan\delta \;=\; \frac{mp\,v^2}{\kappa} \;=\; \frac{p\,v^2}{G_N M_S}\,. \tag{6.64}$$

Here in the second equality we have inserted the value $\kappa = G_N M_S\, m$, where $M_S$ is the mass of the Sun.

The undeflected path in Figure 20 has the comet coming in at an angle $\theta = 0$ (in the limit $R \to \infty$), and going past the Sun to $\theta = \pi$. On the other hand, the actual path sends the comet back out to infinity at an angle $2\pi - 2\delta$. It follows that the comet is deflected through an angle $\pi - 2\delta$. See Figure 21. ■

## 6.3 Kepler's laws

In the late 16th century the Danish nobleman Tycho Brahe made accurate and comprehensive planetary observations, which Johannes Kepler was then able to analyse. Using this empirical data Kepler remarkably deduced the following three laws (published between 1609 and 1619):

**K1**: The path of each planet is an ellipse with the Sun at the focus.

**K2**: A straight line joining the Sun and a planet sweeps out equal areas in equal times.

**K3**: The square of each planet's period is proportional to the cube of the semi-major axis of its elliptical orbit.

The force attracting a planet to the Sun is of course Newton's inverse square law of gravitation, which we solved in the previous subsection. Putting the Sun at the origin, this indeed turns out to be the focus of an ellipse for bounded orbits. As we remarked earlier, just as the Sun attracts the planet, by Newton's third law the planet also attracts the Sun, which hence accelerates and cannot be the origin of an inertial frame. This is true, but the Sun is so much more massive (more than a factor of $10^3$) than any planet that its centre can be taken to be *approximately* fixed. We'll discuss this more carefully in section 7.3. Notice that we are also ignoring the fact that in our solar system there are many planets, which also attract each other – but this is again a subleading effect.

We thus take it as read that we have proven **K1** from Newton's laws. What about **K2** and **K3**?

**Proof of K2**: Kepler's second law is a simple consequence of conservation of angular momentum, expressed through equation (6.31). Recall the latter reads $r^2\dot{\theta} = h =$ constant. A straight line from the Sun to a planet is simply the position vector $\mathbf{r}(t)$. In a small time interval $\delta t$ the planet sweeps out a small triangle with base length $r$ and height $r\delta\theta$ (see Figure 22), which has area $\delta A = \frac{1}{2}r \cdot r\delta\theta = \frac{1}{2}r^2\dot{\theta}\,\delta t$. We thus deduce that

$$\dot{A} \;=\; \frac{1}{2}r^2\dot{\theta} \;=\; \frac{1}{2}h \;=\; \text{constant}. \tag{6.65}$$

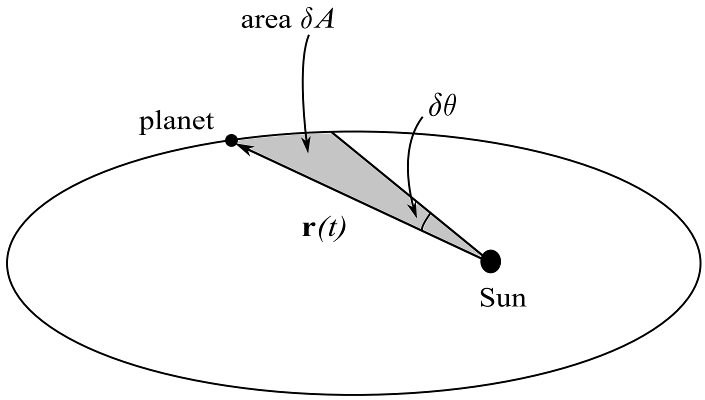



Figure 22: In a small time interval $\delta t$ the angle subtended at the origin changes by a small amount $\delta\theta = \dot{\theta}\,\delta t$, sweeping out an area $\delta A$.

Being a consequence only of conservation of angular momentum, Kepler's second law holds for *any* central force (even non-conservative ones). ■

**Proof of K3**: The third law requires only a little more work. First we recall that the area of an ellipse with semi-major axis $a$ and semi-minor axis $b$ is

$$A = \pi a b \,. \tag{6.66}$$

On the other hand, we know from **K2** that this area is swept out at a constant rate $\dot{A} = \frac{1}{2}h$. Integrating this over one period we obtain

$$A = \int \mathrm{d}A = \frac{1}{2}h \int \mathrm{d}t = \frac{1}{2}hT \,. \tag{6.67}$$

Thus the square of the period $T$ is

$$T^2 = \frac{4A^2}{h^2} = \frac{4\pi^2 a^2 b^2}{h^2} = \frac{4\pi^2}{G_N M_S} \cdot \frac{a^2 b^2}{r_0} \,, \tag{6.68}$$

where in the last step we have substituted $h^2 = \kappa r_0/m = G_N M_S r_0$ using (6.40). Using (6.44) we may tidy up the last geometric factor in (6.68) as $a^2b^2/r_0 = r_0^3/(1-e^2)^3 = a^3$, giving the final formula

$$T^2 = \frac{4\pi^2 a^3}{G_N M_S} \,. \tag{6.69}$$

This is precisely Kepler's third law, where the proportionality factor is $4\pi^2/G_N M_S$, where $M_S =$ mass of the Sun. ■

## 6.4 Coulomb scattering

In the last two subsections we focused on the Kepler problem $F(r) = -\kappa/r^2$ with $\kappa > 0$. For $\kappa < 0$ the inverse square law force is repulsive, rather than attractive. The equations are similar, but the sign difference is important!

The radial equation (6.33) is still

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + u = \frac{\kappa}{mh^2} , \tag{6.70}$$

but now with $\kappa < 0$ it is more convenient to write the solution as

$$u(\theta) = \frac{1}{r_0}(e \cos\theta - 1) , \qquad \text{where} \qquad r_0 = -\frac{mh^2}{\kappa} > 0 . \tag{6.71}$$

Compare with (6.38), and the analysis thereafter. The solutions (6.71) now only make sense for $e > 1$. These are again hyperbolae, but with the *opposite branch* of the hyperbola compared to the attractive case – see Figure 23. The equations (6.43), (6.46) and (6.47) all still hold, but instead of (6.42) we now have $r = e\,x - r_0$. A quick way to see that we pick the opposite branch is from the asymptotes: from (6.71) we see that $r \to \infty$ along the asymptotes for $\cos\theta = +1/e$, whose two solutions $\theta = \pm\theta_0$ now have $\theta_0 = \cos^{-1}(1/e) \in (0, \pi/2)$.

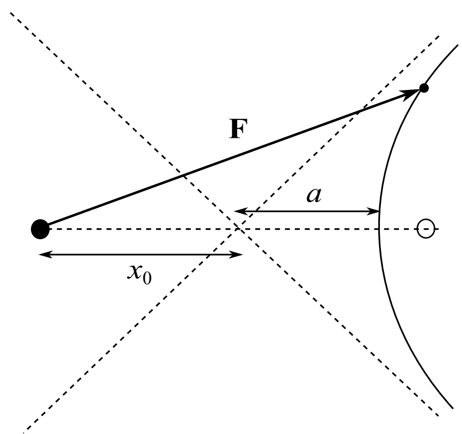



Figure 23: The other branch of the hyperbola in Figure 18b, relevant for the corresponding repulsive force. The large black dot is again the origin, focus and centre of the force.

This $\kappa < 0$ problem describes the classical scattering of a charged particle off another particle with the same sign charge, *e.g.* the scattering of two (negatively charged) electrons, or the scattering of positively charged alpha particles (helium nuclei) off an atomic nucleus. The latter experiment led Rutherford to the correct picture of an atom as a very small positively charged nucleus, containing most of the atomic mass, surrounded by negatively charged electrons.

# 7 Systems of particles

So far we have mainly been studying the motion of a single particle. What can we say about the dynamics of many particles? Before discussing this, now is a good time to discuss the non-uniqueness of inertial frames.

## 7.1 Galilean transformations

As discussed in section 1, in order to apply Newton's second law we must first establish an *inertial* reference frame. Such frames are not unique. Suppose we have an inertial frame $\mathcal{S}$, with respect to which positions are specified by a vector $\mathbf{r} = (x, y, z)$ from the origin $O$. Consider the following transformations to a different frame $\mathcal{S}'$, with positions specified by $\mathbf{r}'$:

$$\begin{cases} \text{spatial translations, } \mathbf{r}' = \mathbf{r} - \mathbf{x}, \text{ where } \mathbf{x} \text{ is a constant vector,} \\ \text{constant rotations, } \mathbf{r}' = \mathcal{R}\,\mathbf{r}, \text{ where } \mathcal{R} \text{ is a constant } 3\times 3 \text{ orthogonal matrix,} \\ \text{Galilean boosts, } \mathbf{r}' = \mathbf{r} - \mathbf{u}\,t, \text{ where } \mathbf{u} \text{ is a constant velocity.} \end{cases}$$

The first and second transformations simply translate the origin by a fixed distance, and rotate the axes by a fixed rotation, respectively. The final transformation has the origins $O$, $O'$ moving at a fixed relative velocity $\mathbf{u}$.

If $\mathbf{r}(t)$ is the trajectory of a *free particle* (by definition, no forces act on it) in the frame $\mathcal{S}$, then $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r} = \mathbf{0}$. It is a simple matter to check that for each $\mathbf{r}'(t)$ above one has $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}' = \mathbf{0}$, and hence the particle also moves with constant velocity in the new frame $\mathcal{S}'$. Any combination of the above transformations thus maps an inertial frame to another inertial frame, generating the *Galilean transformation group*.[16] The freedom to make Galilean transformations is sometimes useful when analysing the dynamics of more than one particle, as we shall see.

The insight of Galileo was that physics is *invariant* under Galilean transformations: the laws of motion are the same in any inertial frame. This is known as *Galileo's principle of relativity*. For example, consider an observer standing on a train moving at constant velocity $\mathbf{u}$, compared to another observer at rest with respect to the Earth. These two inertial frames are in uniform relative motion, so $\mathbf{r}' = \mathbf{r} - (\mathbf{u}\,t + \mathbf{x})$, with $\mathbf{u}$ and $\mathbf{x}$ constant. The laws of motion (Newton's second law) inside the train are not any different from those for the observer at rest. However, as the train turns through a bend in the track a reference frame at rest relative to the train is accelerating, and this is observerd as a "fictitious force" by the passengers inside (luggage falls over, it's less easy to walk down the aisle, *etc*).

## 7.2 Centre of mass motion

**Note on notation**: Henceforth we will *always* denote our inertial frame, in which we write down Newton's second law, as $\hat{\mathcal{S}}$, with origin $\hat{O}$.

[16] Sometimes time translations $t' = t - s$, where $s$ is a constant, are also included in this set of transformations.

Consider a system of $N$ point particles. With respect to an inertial frame $\hat{\mathcal{S}}$, we denote the position vector of the $I$th particle from $\hat{O}$ by $\mathbf{r}_I$, which has mass $m_I$ and hence linear momentum $\mathbf{p}_I = m_I\dot{\mathbf{r}}_I$, $I = 1, \ldots, N$. We suppose that particle $J$ exerts a force $\mathbf{F}_{IJ}$ on particle $I$, for $I \neq J$. Newton's third law immediately tells us that $\mathbf{F}_{JI} = -\mathbf{F}_{IJ}$ for each $I \neq J$. On the other hand Newton's second law for particle $I$ reads

$$m_I\ddot{\mathbf{r}}_I \;=\; \dot{\mathbf{p}}_I \;=\; \mathbf{F}_I \;=\; \mathbf{F}_I^{\text{ext}} + \sum_{J\neq I} \mathbf{F}_{IJ} \; . \tag{7.1}$$

Here we have included an *external force* $\mathbf{F}_I^{\text{ext}}$, *i.e.* a force acting on particle $I$ that is not due to the other $N-1$ particles in the system. We refer to the $\mathbf{F}_{IJ}$ as *internal forces.*

When considering a single particle, the force $\mathbf{F} = \mathbf{F}^{\text{ext}}$ in Newton's second law is by definition always external. In this case we always have in mind that (a) something else is responsible for producing that force, and (b) we are entirely ignoring the effect the particle has on whatever that something else is (*i.e.* we are ignoring its *back-reaction*, under Newton's third law). Of course, whether or not it is reasonable to neglect the effects of the particle on the "external system" that produces $\mathbf{F}^{\text{ext}}$ depends on the circumstances. At the end of section 3.1 we briefly discussed fluid drag in this context. At a molecular level the fluid is made up of fluid particles, and the drag force is an effective external force, resulting from large numbers of collisions of our object with the very small mass fluid particles. It is hardly reasonable to model this by introducing $N \sim 10^{30}$ water molecules! A more subtle example is the Kepler problem in section 6.2. Here a particle of mass $m$ is attracted by an inverse square law force $\mathbf{F} = -(\kappa/r^2)\,\mathbf{e}_r$ directed towards the origin. In this context we were effectively regarding $\mathbf{F}$ as an *external* force acting on the particle, ignoring the fact that the mass at the origin that produces $\mathbf{F}$ will experience an equal and opposite force, and hence accelerate. We'll come back to precisely this point in the next subsection.

**Definition** The *centre of mass* of the system of particles is the point $G$, with position vector

$$\mathbf{R}_G \;\equiv\; \frac{1}{M}\sum_{I=1}^{N} m_I\mathbf{r}_I \; , \tag{7.2}$$

where $M = \sum_{I=1}^{N} m_I$ is the *total mass.* Similarly the *total momentum* of the system is

$$\mathbf{P} \;\equiv\; \sum_{I=1}^{N} \mathbf{p}_I \;=\; M\dot{\mathbf{R}}_G \; . \tag{7.3}$$

The key point of this is the following:

**Theorem** The centre of mass of the system behaves like a point particle of mass $M$ acted on by the *total external force.* In particular, the dynamics of the centre of mass is independent of the internal forces.

**Proof**: We compute

$$M\ddot{\mathbf{R}}_G \;=\; \dot{\mathbf{P}} \;=\; \sum_{I=1}^{N}\dot{\mathbf{p}}_I \;=\; \sum_{I=1}^{N}\left(\mathbf{F}_I^{\rm ext}+\sum_{J\neq I}\mathbf{F}_{IJ}\right) , \tag{7.4}$$

the last equality using (7.1). However, due to Newton's third law $\mathbf{F}_{IJ} = -\mathbf{F}_{JI}$, the $N(N-1)$ terms in the sum

$$\sum_{I=1}^{N}\sum_{J\neq I}\mathbf{F}_{IJ} \;=\; \mathbf{0} \tag{7.5}$$

cancel pairwise. Thus (7.4) becomes

$$M\ddot{\mathbf{R}}_G \;=\; \dot{\mathbf{P}} \;=\; \sum_{I=1}^{N}\mathbf{F}_I^{\rm ext} \;=\; \mathbf{F}^{\rm ext} , \tag{7.6}$$

where $\mathbf{F}^{\rm ext}$ is by definition the total external force. ■

This result explains why we can (often) so accurately model objects as point particles, even when they manifestly are not. Whatever internal forces are acting within our object, for example holding it together, they will cancel out of the centre of mass motion. In most of the problems we have studied we have then really been modelling the centre of mass motion of an object, and we've been applying Newton's second law in the form (7.6).

**Definition** A *closed system* is one in which all forces are internal, acting between the constituents of the system. That is, $\mathbf{F}_I^{\rm ext} = \mathbf{0}$, $I = 1, \ldots, N$.

We then have the following important corollary:

**Corollary** For a closed system the total momentum is conserved, $\dot{\mathbf{P}} = \mathbf{0}$.

This is of course an immediate consequence of (7.6). When the total momentum is conserved notice that the centre of mass moves with constant velocity $\dot{\mathbf{R}}_G =$ constant. This means that by a suitable *Galilean transformation* (a Galilean boost and translation) we may take the centre of mass to be $\mathbf{R}_G = \mathbf{0}$, the origin of our inertial frame.

**Definition** For a system with $\mathbf{F}^{\rm ext} = \mathbf{0}$, the inertial frame in which the centre of mass $\mathbf{R}_G = \mathbf{0}$ is called the *centre of mass frame.* (This is unique up to an overall constant rotation of the axes.)

**Definition** The *total angular momentum* $\mathbf{L} = \mathbf{L}_P$ of the system about a point $P$ is

$$\mathbf{L}_P \;=\; \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x})\wedge\mathbf{p}_I , \tag{7.7}$$

where $P$ has position vector $\mathbf{x}$ from the origin $\hat{O}$. That is, $\mathbf{L}$ is the vector sum of the angular momenta $\mathbf{L}_I = (\mathbf{r}_I - \mathbf{x})\wedge\mathbf{p}_I$ for each particle $I$ about $P$ – see (4.17).

Using the definition (7.7) we begin by computing

$$\begin{aligned}\dot{\mathbf{L}}_P &= \sum_{I=1}^{N}[(\dot{\mathbf{r}}_I - \dot{\mathbf{x}}) \wedge \mathbf{p}_I + (\mathbf{r}_I - \mathbf{x}) \wedge \dot{\mathbf{p}}_I] \\ &= -\dot{\mathbf{x}} \wedge \mathbf{P} + \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \dot{\mathbf{p}}_I \\ &= -\dot{\mathbf{x}} \wedge \mathbf{P} + \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \Big(\mathbf{F}_I^{\mathrm{ext}} + \sum_{J\neq I}\mathbf{F}_{IJ}\Big) .\end{aligned} \tag{7.8}$$

Here in the second equality we have used $\dot{\mathbf{r}}_I \wedge \mathbf{p}_I = \dot{\mathbf{r}}_I \wedge m_I \dot{\mathbf{r}}_I = \mathbf{0}$. The third equality uses Newton's second law (7.1). In $\sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \sum_{J\neq I}\mathbf{F}_{IJ}$ we again have $\frac{1}{2}N(N-1)$ pairs of terms, which look like

$$(\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{F}_{IJ} + (\mathbf{r}_J - \mathbf{x}) \wedge \mathbf{F}_{JI} \;=\; (\mathbf{r}_I - \mathbf{r}_J) \wedge \mathbf{F}_{IJ} , \tag{7.9}$$

and we have used Newton's third law. To get any further we need the *strong form* of Newton's third law:

> **N3** (strong form): If particle 1 exerts a force $\mathbf{F} = \mathbf{F}_{21}$ on particle 2, then particle 2 also exerts a force $\mathbf{F}_{12} = -\mathbf{F}$ on particle 1. Moreover, this force acts along the vector connecting particle 1 to particle 2, $\mathbf{F}_{12} \propto (\mathbf{r}_1 - \mathbf{r}_2)$.

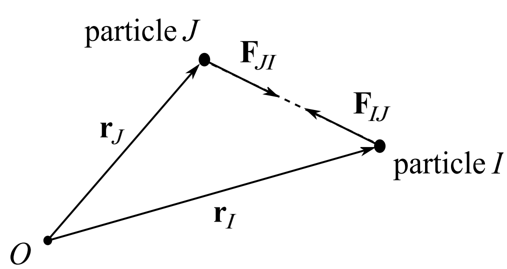



Figure 24: The strong form of Newton's third law.

This clearly holds for the inverse square law forces of Newton (6.5) and Coulomb (6.19), but there are examples that don't satisfy this.[17]

Returning to (7.9), we see that if the strong form of Newton's third law holds this is zero, and hence (7.8) gives

$$\dot{\mathbf{L}}_P \;=\; -\dot{\mathbf{x}} \wedge \mathbf{P} + \sum_{I=1}^{N}(\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{F}_I^{\mathrm{ext}} \;=\; -\dot{\mathbf{x}} \wedge \mathbf{P} + \boldsymbol{\tau}_P^{\mathrm{ext}} , \tag{7.10}$$

[17]** Notably the magnetic component of the Lorentz force acting on a charged particle moving in the electromagnetic field generated by another charged particle. But explaining this is well beyond our syllabus!

where $\boldsymbol{\tau}_P^{\text{ext}}$ is by definition the *total external torque* about $P$, *c.f.* (4.22).

There are two special cases where the first term on the right hand side of (7.10) is zero: (i) taking $P = \hat{O}$ gives $\mathbf{x} = \mathbf{0}$, (ii) taking instead $P = G$ we have $\dot{\mathbf{x}} \wedge \mathbf{P} = \dot{\mathbf{R}}_G \wedge \mathbf{P} = \dot{\mathbf{R}}_G \wedge M\dot{\mathbf{R}}_G = \mathbf{0}$. We have thus proven:

**Theorem** Provided the strong form of Newton's third law holds, the rate of change of total angular momentum about either $\hat{O}/G$ equals the total external torque about $\hat{O}/G$. That is,

$$\dot{\mathbf{L}}_{\hat{O}/G} \quad = \quad \boldsymbol{\tau}^{\text{ext}}_{\hat{O}/G} \, . \tag{7.11}$$

**Corollary** For a closed system satisfying the strong form of Newton's third law, the total angular momentum about the origin $\hat{O}$ of an inertial frame is conserved, $\dot{\mathbf{L}}_{\hat{O}} = \mathbf{0}$.

The main application of (7.11) will be to rigid body motion, which is the subject of section 8.3. In particular the following result will be useful:

**Proposition** Consider the system of particles in a uniform gravitational field, with acceleration due to gravity $g$. Assuming this is the only external force acting, the total external torque about a point $P$ with position vector $\mathbf{x}$ is

$$\boldsymbol{\tau}_P^{\text{ext}} \quad = \quad -(\mathbf{R}_G - \mathbf{x}) \wedge Mg\,\mathbf{k} \, . \tag{7.12}$$

This is the same as the torque for a particle of mass $M$ at the centre of mass $\mathbf{R}_G$ (compare to (4.22)). In particular, the torque about $G$ (for which $\mathbf{x} = \mathbf{R}_G$) is zero.

**Proof**: We simply compute

$$\boldsymbol{\tau}_P^{\text{ext}} \quad \equiv \quad \sum_{I=1}^{N} (\mathbf{r}_I - \mathbf{x}) \wedge \mathbf{F}_I^{\text{ext}} \quad = \quad \sum_{I=1}^{N} (\mathbf{r}_I - \mathbf{x}) \wedge (-m_I g\,\mathbf{k}) \quad = \quad -(\mathbf{R}_G - \mathbf{x}) \wedge Mg\,\mathbf{k} \, , \tag{7.13}$$

where we have used the definitions $M = \sum_{I=1}^{N} m_I$, $M\mathbf{R}_G = \sum_{I=1}^{N} m_I \mathbf{r}_I$ in the final equality. ■

## 7.3 The two-body problem

A closed system with a single point particle isn't very interesting: there is no force acting, and the particle moves with constant momentum. The *two-body problem* is a closed system of two point particles. Newton's second and third laws give

$$m_1\ddot{\mathbf{r}}_1 \quad = \quad \mathbf{F}_{12} \, , \qquad m_2\ddot{\mathbf{r}}_2 \quad = \quad \mathbf{F}_{21} \quad = \quad -\mathbf{F}_{12} \, . \tag{7.14}$$

Adding these two equations implies that the centre of mass

$$\mathbf{R}_G \quad = \quad \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2} \tag{7.15}$$

moves with constant velocity, which we knew from the last subsection. On the other hand, if we define $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$ so that

$$\mathbf{r}_1 = \mathbf{R}_G + \frac{m_2}{m_1+m_2}\mathbf{r} \ , \qquad \mathbf{r}_2 = \mathbf{R}_G - \frac{m_1}{m_1+m_2}\mathbf{r} \ , \tag{7.16}$$

then from (7.14) we deduce

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{12} = \frac{m_1+m_2}{m_1 m_2}\mathbf{F}_{12} \ . \tag{7.17}$$

**Definition** The *reduced mass* for the two-body problem is $\mu = \dfrac{m_1 m_2}{m_1+m_2}$.

In terms of this the equation of motion (7.17) reads

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{12} \ . \tag{7.18}$$

**Example**: For the inverse square law force $\mathbf{F}_{12} = -\dfrac{\kappa}{|\mathbf{r}_1 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) = -\dfrac{\kappa}{r^2}\dfrac{\mathbf{r}}{r}$, where $r = |\mathbf{r}|$.

We have thus *effectively* reduced the two-body problem to a problem for a *single* particle, with position vector $\mathbf{r}(t)$ satisfying (7.18). The force on the right hand side is then *effectively* an external force for this particle. Having solved this, the solution to the original two-body problem is given by (7.16). In fact we are always free to define $\mathbf{r}_1^* = \mathbf{r}_1 - \mathbf{R}_G$, $\mathbf{r}_2^* = \mathbf{r}_2 - \mathbf{R}_G$, which are the positions of the two particles in the *centre of mass frame*. Recall this is a Galilean transformation, since $\dot{\mathbf{R}}_G =$ constant. In this inertial frame (7.16) becomes

$$\mathbf{r}_1^* = \frac{m_2}{m_1+m_2}\mathbf{r} \ , \qquad \mathbf{r}_2^* = -\frac{m_1}{m_1+m_2}\mathbf{r} \ . \tag{7.19}$$

If the second mass is much larger than the first, $m_2 \gg m_1$, then these become $\mathbf{r}_1^* \simeq \mathbf{r}$, $\mathbf{r}_2^* \simeq \mathbf{0}$, while the reduced mass is $\mu \simeq m_1$. We may thus view what we did in solving the Kepler problem in section 6.2 in two different ways:

- If we take the mass $m = \mu$ in (6.27), then in section 6.2 we were really solving (7.18) for the two-body problem. This describes the exact internal relative motion of the two bodies.

- If we instead take the mass $m_2 \gg m_1$ then the solution in section 6.2 is the *approximate* solution to the two-body problem in the centre of mass frame, where the larger mass $m_2$ is at the origin.

Usually the latter is applicable, *e.g.* the Sun is more than 1000 times more massive than any of the planets, while for a satellite or comet orbiting the Earth the factor is many orders of magnitude larger still. What's remarkable about the two-body problem is that the exact solution and approximate solution we have described are mathematically equivalent, differing only in which mass to use in Newton's second law!

# 8 Rotating frames and rigid bodies

In this final section we discuss two topics that involve rotation: the dynamics of *rigid bodies* in sections 8.2 and 8.3, and Newton's laws in a general (*i.e.* non-inertial) frame from section 8.4 to the end. We will only describe the basic features of rigid body motion, focusing on simple examples; a general discussion is left to B7.1 Classical Mechanics (which also introduces and exploits more powerful methods for solving the dynamics).

## 8.1 Rotating frames

Throughout this section there will always be *two* reference frames in the problem, and it is important to make clear which is which from the outset:

> A fixed inertial frame $\hat{\mathcal{S}}$: this has origin $\hat{O}$ and fixed coordinate axes with corresponding basis vectors $\hat{\mathbf{e}}_i$, $i = 1, 2, 3$.
>
> A general frame $\mathcal{S}$: this has origin $O$, with position vector $\mathbf{x}$ as measured from $\hat{O}$, and coordinate axes with corresponding basis vectors $\mathbf{e}_i$, $i = 1, 2, 3$.

Whenever we write down Newton's laws of motion, we must do so using the inertial frame $\hat{\mathcal{S}}$. This is the frame of an inertial observer, often called the *laboratory frame* by physicists, and we regard it as *fixed* and *time-independent*. In particular this means that the basis vectors $\hat{\mathbf{e}}_i$ are independent of time, $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{e}}_i = \mathbf{0}$, $i = 1, 2, 3$. When we introduce rigid bodies, the frame $\mathcal{S}$ will rotate with the body, and hence will in general be non-inertial.

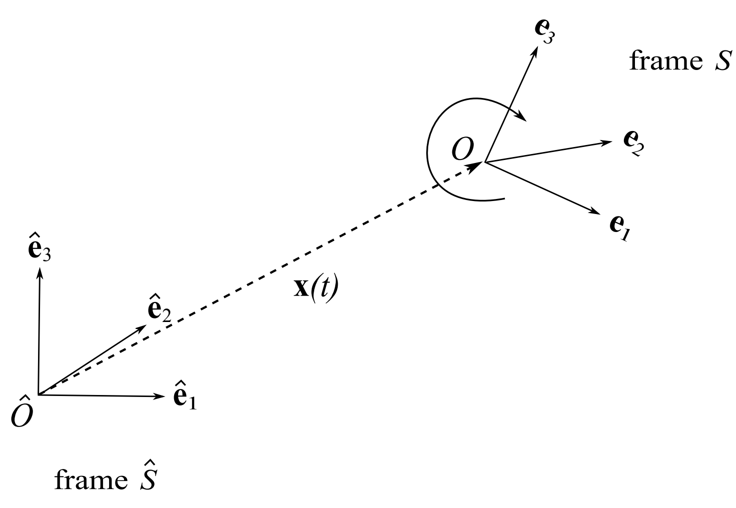



Figure 25: The reference frame $\hat{\mathcal{S}}$ is a fixed inertial frame. This is the frame in which we formulate Newton's laws of motion, and hence do the "observing". With respect to this frame, a general frame $\mathcal{S}$ has origin $O$ at position vector $\mathbf{x} = \mathbf{x}(t)$ as measured from $\hat{O}$, and its coordinate axes may be rotating, so that $\mathbf{e}_i = \mathbf{e}_i(t)$.

We may write the orthonormal basis vectors $\{\mathbf{e}_i\}$ of the frame $\mathcal{S}$ as

$$\mathbf{e}_i(t) \;=\; \sum_{j=1}^{3} \mathcal{R}_{ij}(t)\,\hat{\mathbf{e}}_j\;, \qquad i = 1,2,3\;. \tag{8.1}$$

As you learned in the Geometry course, the fact that both bases are *orthonormal* means that $\mathcal{R} = (\mathcal{R}_{ij})$ is an *orthogonal matrix*. The main result of this subsection is:

**Proposition** There is a (unique) vector $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ such that

$$\dot{\mathbf{e}}_i \;=\; \boldsymbol{\omega} \wedge \mathbf{e}_i\;, \qquad i = 1,2,3\;. \tag{8.2}$$

$\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ is called the *angular velocity* of the frame $\mathcal{S}$ with respect to fixed inertial frame $\hat{\mathcal{S}}$.

We give two proofs below. The first doesn't use orthogonal matrices directly, while the second does. (The second proof appeared in the Geometry course.)

**Proof 1**: The fact that $\{\mathbf{e}_i\}$ is an orthonormal basis means that

$$\mathbf{e}_i \cdot \mathbf{e}_j \;=\; \delta_{ij} \;=\; \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}\;. \tag{8.3}$$

Differentiating this with respect to time $t$ thus gives

$$\dot{\mathbf{e}}_i \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \dot{\mathbf{e}}_j \;=\; 0\;. \tag{8.4}$$

In particular taking $i = j$ gives the three equations $\dot{\mathbf{e}}_1 \cdot \mathbf{e}_1 = \dot{\mathbf{e}}_2 \cdot \mathbf{e}_2 = \dot{\mathbf{e}}_3 \cdot \mathbf{e}_3 = 0$. This means that $\dot{\mathbf{e}}_i$ is orthogonal to $\mathbf{e}_i$ for each $i = 1,2,3$, and hence we may write

$$\dot{\mathbf{e}}_1 \;=\; \gamma\,\mathbf{e}_2 - \beta\,\mathbf{e}_3\;, \qquad \dot{\mathbf{e}}_2 \;=\; \alpha\,\mathbf{e}_3 - \lambda\,\mathbf{e}_1\;, \qquad \dot{\mathbf{e}}_3 \;=\; \nu\,\mathbf{e}_1 - \mu\,\mathbf{e}_2\;, \tag{8.5}$$

where $\alpha, \beta, \gamma, \lambda, \mu$ and $\nu$ are functions of time $t$. The remaining content of (8.4) gives

$$\begin{aligned} \dot{\mathbf{e}}_1 \cdot \mathbf{e}_2 + \mathbf{e}_1 \cdot \dot{\mathbf{e}}_2 \;&=\; 0 \qquad \Longrightarrow \qquad \lambda \;=\; \gamma\;, \\ \dot{\mathbf{e}}_2 \cdot \mathbf{e}_3 + \mathbf{e}_2 \cdot \dot{\mathbf{e}}_3 \;&=\; 0 \qquad \Longrightarrow \qquad \mu \;=\; \alpha\;, \\ \dot{\mathbf{e}}_3 \cdot \mathbf{e}_1 + \mathbf{e}_3 \cdot \dot{\mathbf{e}}_1 \;&=\; 0 \qquad \Longrightarrow \qquad \nu \;=\; \beta\;. \end{aligned} \tag{8.6}$$

It follows that

$$\dot{\mathbf{e}}_1 \;=\; \gamma\,\mathbf{e}_2 - \beta\,\mathbf{e}_3\;, \qquad \dot{\mathbf{e}}_2 \;=\; \alpha\,\mathbf{e}_3 - \gamma\,\mathbf{e}_1\;, \qquad \dot{\mathbf{e}}_3 \;=\; \beta\,\mathbf{e}_1 - \alpha\,\mathbf{e}_2\;, \tag{8.7}$$

which may be written more succinctly as (8.2), where $\boldsymbol{\omega} = (\alpha, \beta, \gamma) = \alpha\,\mathbf{e}_1 + \beta\,\mathbf{e}_2 + \gamma\,\mathbf{e}_3$, and we have used $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_3$, plus the cyclic permutations $\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e}_3 \wedge \mathbf{e}_1 = \mathbf{e}_2$. ■

**Proof 2**: The alternative proof of this Proposition instead takes the time derivative of (8.1):

$$\dot{\mathbf{e}}_i \;=\; \sum_{j=1}^{3} \dot{\mathcal{R}}_{ij}\,\hat{\mathbf{e}}_j \;=\; \sum_{j,k=1}^{3} \dot{\mathcal{R}}_{ij}\mathcal{R}_{kj}\,\mathbf{e}_k \;=\; \sum_{k=1}^{3} (\dot{\mathcal{R}}\,\mathcal{R}^{\mathrm{T}})_{ik}\,\mathbf{e}_k\;, \tag{8.8}$$

where in the second equality we have used the fact that $\mathcal{R}$ is orthogonal, and hence $\mathcal{R}^{-1} = \mathcal{R}^{\mathrm{T}}$. In the Geometry course the angular velocity vector $\boldsymbol{\omega}$ was instead introduced by noting that $(\dot{\mathcal{R}}\,\mathcal{R}^{\mathrm{T}})$ is an anti-symmetric matrix, so that we can write

$$\dot{\mathcal{R}}\,\mathcal{R}^{\mathrm{T}} = \begin{pmatrix} 0 & \gamma & -\beta \\ -\gamma & 0 & \alpha \\ \beta & -\alpha & 0 \end{pmatrix} . \tag{8.9}$$

Then (8.8) is equivalent to (8.2) with $\boldsymbol{\omega} = (\alpha, \beta, \gamma)$ – to see this just look at (8.7): the three rows of (8.9) give the right hand sides of the three equations in (8.7), respectively. ■

**Example**: Consider the special case in which

$$\mathcal{R}(t) = \begin{pmatrix} \cos\theta(t) & \sin\theta(t) & 0 \\ -\sin\theta(t) & \cos\theta(t) & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{8.10}$$

so that $\mathbf{e}_1 = \cos\theta(t)\,\hat{\mathbf{e}}_1 + \sin\theta(t)\,\hat{\mathbf{e}}_2$, $\mathbf{e}_2 = -\sin\theta(t)\,\hat{\mathbf{e}}_1 + \cos\theta(t)\,\hat{\mathbf{e}}_2$, $\mathbf{e}_3 = \hat{\mathbf{e}}_3$. This is a rotation about the third axis $\hat{\mathbf{e}}_3 = \mathbf{e}_3$ by an angle $\theta = \theta(t)$. Then we compute $\dot{\mathbf{e}}_1 = \dot{\theta}\,\mathbf{e}_2$, $\dot{\mathbf{e}}_2 = -\dot{\theta}\,\mathbf{e}_1$, $\dot{\mathbf{e}}_3 = \mathbf{0}$, and hence from (8.2) that $\boldsymbol{\omega} = \dot{\theta}\,\mathbf{e}_3$. ■

We can gain some more intuition for the formula (8.2) by thinking about the position vector $\mathbf{r}$ of a particle. Suppose for simplicity that the two origins coincide (for all time), so that $O = \hat{O}$. We have two bases, $\{\mathbf{e}_i\}$ and $\{\hat{\mathbf{e}}_i\}$, and we may expand the same vector $\mathbf{r}$ in both bases as $\mathbf{r} = \sum_{i=1}^3 r_i\,\mathbf{e}_i = \sum_{i=1}^3 \hat{r}_i\,\hat{\mathbf{e}}_i$. Here $r_i$ are the *components of* $\mathbf{r}$ *in the frame* $\mathcal{S}$, while $\hat{r}_i$ are the components in the frame $\hat{\mathcal{S}}$. In section 1.1 we would have referred to the position vector in the two frames as $\mathbf{r}$ and $\hat{\mathbf{r}}$, respectively, because we wanted to emphasize that it is the *components* $r_i$ and $\hat{r}_i$ that we measure in the frames. However, $\mathbf{r}$ and $\hat{\mathbf{r}}$ are the same vector, just expressed in different bases. The velocity of the particle in the inertial frame $\hat{\mathcal{S}}$ is

$$\begin{aligned} \dot{\mathbf{r}} &= \sum_{i=1}^{3} \dot{r}_i\,\mathbf{e}_i + \sum_{i=1}^{3} r_i\,\dot{\mathbf{e}}_i = \sum_{i=1}^{3} \dot{r}_i\,\mathbf{e}_i + \sum_{i=1}^{3} r_i\,\boldsymbol{\omega}\wedge\mathbf{e}_i \\ &= \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\mathbf{r} . \end{aligned} \tag{8.11}$$

Here we have introduced:

**Definition** The *time derivative* of $\mathbf{r} = \mathbf{r}(t)$ in the frame $\mathcal{S}$ is $\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} \equiv \dot{r}_1\,\mathbf{e}_1 + \dot{r}_2\,\mathbf{e}_2 + \dot{r}_3\,\mathbf{e}_3$. That is, we simply differentiate the *components* of $\mathbf{r}$ in the orthonormal basis $\{\mathbf{e}_i\}$ for $\mathcal{S}$.

We should then never simply write "$\dot{\mathbf{r}}$" when there are two general reference frames being used, because whether or not something is moving depends on who is doing the measuring. However, when we *do* write "$\dot{\mathbf{r}}$" we will always mean the time derivative in the *inertial frame* $\hat{\mathcal{S}}$. Then (8.11) more properly reads

**Proposition** (The Coriolis formula)

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}} \;=\; \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\mathbf{r}\;, \tag{8.12}$$

where $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$.

For rigid body dynamics we will be interested in the velocity of points $\mathbf{r}$ that are *fixed relative to the rotating frame* $\mathcal{S}$. By definition this means that the first term on the right hand side of (8.12) is zero, and hence we may simply write

$$\dot{\mathbf{r}} \;=\; \boldsymbol{\omega}\wedge\mathbf{r}\;. \tag{8.13}$$

To get some geometric intuition for this, consider the change $\delta\mathbf{r}$ in $\mathbf{r}$ in a small time interval $\delta t$. This is $\delta\mathbf{r} = \boldsymbol{\omega}\wedge\mathbf{r}\,\delta t$. By definition of the cross product, this vector is orthogonal to both $\boldsymbol{\omega}$ and $\mathbf{r}$, and has modulus $|\delta\mathbf{r}| = |\mathbf{r}|\sin\alpha\cdot|\boldsymbol{\omega}|\,\delta t$, where $\alpha$ is the angle between $\boldsymbol{\omega}$ and $\mathbf{r}$ – see Figure 26. As seen in $\hat{\mathcal{S}}$, the change $\delta\mathbf{r}$ in the position vector $\mathbf{r}$ of a point fixed in the frame $\mathcal{S}$ in the time interval $\delta t$ is hence obtained by rotating $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\,\delta t$ about the axis parallel to the vector $\boldsymbol{\omega}$.

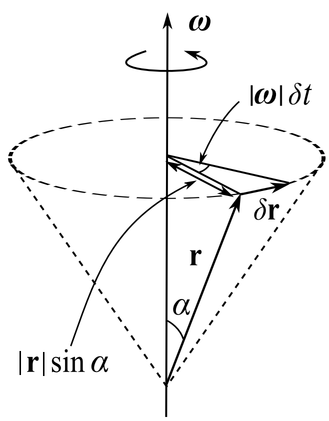



Figure 26: As seen in the inertial frame $\hat{\mathcal{S}}$, the position vector $\mathbf{r}$ of a point $P$ fixed in the frame $\mathcal{S}$ changes by $\delta\mathbf{r} = \boldsymbol{\omega}\wedge\mathbf{r}\,\delta t$ in a small time interval $\delta t$. This is a rotation of $\mathbf{r}$ through an angle $|\boldsymbol{\omega}|\delta t$ about an axis parallel to the vector $\boldsymbol{\omega}$. The direction of rotation is given by the right hand rule.

The above picture leads to the following:

**Definition** In general we may write $\boldsymbol{\omega} = \omega\,\mathbf{n}$, where $\omega = \omega(t) = |\boldsymbol{\omega}|$ is the *angular speed*, and $\mathbf{n} = \mathbf{n}(t)$ is the *instantaneous axis of rotation.*

## 8.2 Rigid bodies

A *rigid body* may be defined as any distribution of mass for which the distance between any two points is fixed. A simple model for this is to take a finite number of point particles, as in section 7.2, but with the constraint that the position vectors $\mathbf{r}_I$ $(I = 1, \ldots, N)$ satisfy $|\mathbf{r}_I - \mathbf{r}_J| = c_{IJ} = \text{constant}$. This ensures that the body retains its size, shape and distribution of mass. One might imagine the $\mathbf{r}_I$ as the positions of atoms in a solid, with the constraints arising from inter-molecular forces. We assume these constraint forces satisfy the strong form of Newton's third law. For now we will stick with this point particle model, but later we will model a rigid body as a continuous distribution of matter, which may be regarded as a limit of the point particle model in which the number of particles tends to infinity.

Choose a point $O$ that is fixed in the body. For example, in the point particle model this could be one of the particles, although as we shall see below it will often be convenient to take this to be the *centre of mass*. We denote the position vector of $O$ as $\mathbf{x} = \mathbf{x}(t)$, where this is measured from the origin $\hat{O}$ of the inertial frame $\hat{\mathcal{S}}$. We may then write

$$\mathbf{R}_I = \mathbf{x} + \mathbf{r}_I \ , \qquad I = 1, \ldots, N \ , \tag{8.14}$$

so that $\mathbf{R}_I$ and $\mathbf{r}_I$ are the positions of the body particles, as measured from $\hat{O}$ and $O$, respectively. See Figure 27.

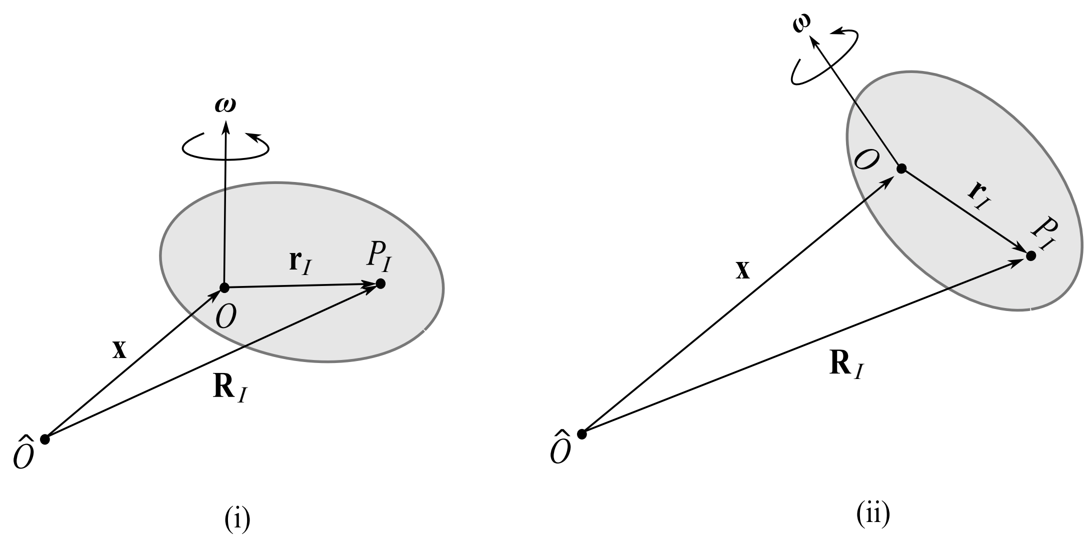



Figure 27: We fix a point $O$ in the rigid body, which is taken to be the origin of the rest frame $\mathcal{S}$ of the body. The frame $\mathcal{S}$ has angular velocity $\boldsymbol{\omega}$, and its origin $O$ has position vector $\mathbf{x}$ relative to the origin $\hat{O}$ of an inertial frame $\hat{\mathcal{S}}$. The body particles $P_I$ have position vectors $\mathbf{r}_I$, measured from $O$. Figures (i) and (ii) show the same body at two different times.

**Definition** The *rest frame* $\mathcal{S}$ of the rigid body is a reference frame, with origin $O$, with respect to which the $\mathbf{r}_I$ are fixed (at rest), *i.e.* $\left(\frac{\mathrm{d}\mathbf{r}_I}{\mathrm{d}t}\right)_{\mathcal{S}} = \mathbf{0}$ for all $I = 1, \ldots, N$.

The existence of such a frame is really equivalent to what we mean by a rigid body in the first place. Provided the matter distribution is not all along a line, the rest frame is defined uniquely by the body, up to a *constant* rotation of its axes and a translation of the origin by a *constant* vector (relative to $\mathcal{S}$).

Using the Coriolis formula (8.12) we then have the important result that

$$\dot{\mathbf{R}}_I \;=\; \dot{\mathbf{x}} + \dot{\mathbf{r}}_I \;=\; \mathbf{v}_O + \boldsymbol{\omega} \wedge \mathbf{r}_I \;. \tag{8.15}$$

Here $\mathbf{v}_O = \dot{\mathbf{x}}$ is the velocity of $O$, as measured in the inertial frame $\hat{\mathcal{S}}$, while $\boldsymbol{\omega}$ is the angular velocity of the rest frame $\mathcal{S}$ with respect to $\hat{\mathcal{S}}$.

As we already mentioned, a natural choice for $O$ is the centre of mass $G$ of the body. This means that $\mathbf{x} = \mathbf{R}_G$, in the notation of section 7.2.[18] From (7.2) and (8.14) we have

$$\mathbf{R}_G \;=\; \frac{1}{M}\sum_{I=1}^{N} m_I \mathbf{R}_I \;=\; \frac{1}{M}\sum_{I=1}^{N} m_I(\mathbf{R}_G + \mathbf{r}_I) \;=\; \mathbf{R}_G + \frac{1}{M}\sum_{I=1}^{N} m_I \mathbf{r}_I \;, \tag{8.16}$$

which implies the constraint

$$\sum_{I=1}^{N} m_I \mathbf{r}_I \;=\; \mathbf{0} \tag{8.17}$$

on the $\mathbf{r}_I$. Let's re-examine the formulas for the total momentum (linear and angular) from section 7.2, and also look at the total kinetic energy. We take $O = G$, unless otherwise stated.

**Linear momentum**

We already know from (7.3) that $\mathbf{P} = M\dot{\mathbf{R}}_G = M\mathbf{v}_G$, but it's interesting to see this explicitly in our current set up:

$$\mathbf{P} \;=\; \sum_{I=1}^{N} m_I \dot{\mathbf{R}}_I \;=\; \sum_{I=1}^{N} m_I(\dot{\mathbf{R}}_G + \boldsymbol{\omega}\wedge\mathbf{r}_I) \;=\; M\dot{\mathbf{R}}_G + \boldsymbol{\omega}\wedge\left(\sum_{I=1}^{N} m_I\mathbf{r}_I\right) \;=\; M\dot{\mathbf{R}}_G \;. \tag{8.18}$$

Here we've used equation (8.15) with $\mathbf{x} = \mathbf{R}_G$ in the second equality, while the last equality uses the constraint (8.17). The total momentum is hence as if the whole mass $M$ were concentrated at the centre of mass $G$.

**Angular momentum**

The total angular momentum *about the centre of mass* $O = G$ is by definition

$$\mathbf{L}_G \;=\; \sum_{I=1}^{N} \mathbf{r}_I \wedge m_I \dot{\mathbf{R}}_I \;=\; \sum_{I=1}^{N} m_I \mathbf{r}_I \wedge (\dot{\mathbf{R}}_G + \boldsymbol{\omega}\wedge\mathbf{r}_I) \;=\; \sum_{I=1}^{N} m_I \mathbf{r}_I \wedge (\boldsymbol{\omega}\wedge\mathbf{r}_I) \;. \tag{8.19}$$

We emphasize again that we have chosen to compute the angular momentum about $O = G$, *not* about the origin $\hat{O}$ of the inertial frame. That latter would be $\mathbf{L}_{\hat{O}}$, and have $\mathbf{R}_I$ in place of $\mathbf{r}_I$ after the first equals sign. The last equality follows from the constraint (8.17). Using the vector identity $\mathbf{r}_I \wedge (\boldsymbol{\omega}\wedge\mathbf{r}_I) = (\mathbf{r}_I\cdot\mathbf{r}_I)\boldsymbol{\omega} - (\mathbf{r}_I\cdot\boldsymbol{\omega})\mathbf{r}_I$, we may hence write

$$\mathbf{L}_G \;=\; \sum_{I=1}^{N} m_I \left[(\mathbf{r}_I\cdot\mathbf{r}_I)\boldsymbol{\omega} - (\mathbf{r}_I\cdot\boldsymbol{\omega})\mathbf{r}_I\right] \;. \tag{8.20}$$

[18] Notice that in this section we are denoting the positions of the particles measured from the origin $\hat{O}$ of the inertial frame by $\mathbf{R}_I$. Thus one should replace $\mathbf{r}_I$ by $\mathbf{R}_I$ in the formulas in section 7.2.

**Definition** The *inertia tensor* $\mathcal{I} = \mathcal{I}^{(O)} = (\mathcal{I}_{ij}^{(O)})$ of the rigid body, about a point $O$ fixed in the body, is defined as

$$\mathcal{I}_{ij} \quad = \quad \sum_{I=1}^{N} m_I \left[ (\mathbf{r}_I \cdot \mathbf{r}_I)\delta_{ij} - r_{I\,i}\, r_{I\,j} \right] \ . \tag{8.21}$$

Here $\mathbf{r}_I = \sum_{i=1}^3 r_{I\,i}\, \mathbf{e}_i$ are the position vectors of the body particles, in the rest frame basis $\{\mathbf{e}_i\}$.

Notice the inertia tensor is defined in the rest frame of the body, and so is intrinsic to the body itself, and in particular independent of time $t$. It is also manifestly symmetric, $\mathcal{I} = \mathcal{I}^{\mathrm{T}}$. Note also that the definition depends on a choice of origin $O$, fixed in the body. The point of the definition is that we may now write the total angular momentum (8.20) in matrix notation as

$$\mathbf{L}_G \quad = \quad \mathcal{I}^{(G)}\, \boldsymbol{\omega} \quad = \quad \sum_{i,j=1}^{3} \mathcal{I}_{ij}^{(G)} \omega_j\, \mathbf{e}_i \ . \tag{8.22}$$

**Kinetic energy**

The total kinetic energy of the body, as measured in the inertial frame, is

$$T \quad = \quad \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{R}}_I|^2 \quad = \quad \frac{1}{2} \sum_{I=1}^{N} m_I \left[ |\dot{\mathbf{R}}_G|^2 + 2\dot{\mathbf{R}}_G \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_I) + (\boldsymbol{\omega} \wedge \mathbf{r}_I) \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_I) \right] \ . \tag{8.23}$$

The middle term on the right hand side is again zero, using the constraint (8.17). On the other hand we may rewrite the last term using the vector identify

$$(\boldsymbol{\omega} \wedge \mathbf{r}_I) \cdot (\boldsymbol{\omega} \wedge \mathbf{r}_I) \quad = \quad \boldsymbol{\omega} \cdot (\mathbf{r}_I \wedge (\boldsymbol{\omega} \wedge \mathbf{r}_I)) \ . \tag{8.24}$$

Recalling the formula (8.20), we have thus shown that

$$T \quad = \quad \frac{1}{2} M |\dot{\mathbf{R}}_G|^2 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}_G \ . \tag{8.25}$$

That is, the total kinetic energy is the sum of two terms: the first is due to the centre of mass motion relative to $\hat{O}$, and is again is as though all the mass was concentrated at the centre of mass. The second term is the *rotational kinetic energy* about $G$.

**Definition** The *rotational kinetic energy* about the centre of mass $G$ is

$$T_{\mathrm{rot}} \quad = \quad \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}_G \quad = \quad \frac{1}{2} \boldsymbol{\omega}^{\mathrm{T}} \mathcal{I}^{(G)} \boldsymbol{\omega} \quad = \quad \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{I}_{ij}^{(G)}\, \omega_i\, \omega_j \ . \tag{8.26}$$

These general formulae are very pretty, but they are also quite abstract. We conclude this subsection with some simple examples. Here we usually have in mind a *continuous* distribution of matter, rather than a point particle model. This assumes the distribution of mass in the body is defined by a density $\rho(\mathbf{r})$, so that the mass $\delta m$ in a small volume $\mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ centred at $\mathbf{r} = (r_1, r_2, r_3) = (x, y, z)$

| Linear motion | Angular (rotational) motion |
|---|---|
| Mass $M$ | Inertia tensor $\mathcal{I}$ = "rotational mass" |
| Linear velocity $\dot{\mathbf{R}}$ | Angular velocity $\boldsymbol{\omega}$ |
| Linear speed $\lvert\dot{\mathbf{R}}\rvert$ | Angular speed $\omega = \lvert\boldsymbol{\omega}\rvert$ |
| Linear momentum $\mathbf{P} = M\dot{\mathbf{R}}$ | Angular momentum $\mathbf{L} = \mathcal{I}\,\boldsymbol{\omega}$ |
| Kinetic energy $\frac{1}{2}M\lvert\dot{\mathbf{R}}\rvert^2$ | Rotational kinetic energy $\frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathcal{I}\,\boldsymbol{\omega}$ |
| Equation of motion: $\dot{\mathbf{P}} = \mathbf{F}^{\mathrm{ext}}$ | Angular equation of motion $\dot{\mathbf{L}} = \boldsymbol{\tau}^{\mathrm{ext}}$ |

Table 1: Contrasting linear motion with angular (rotational) motion. Each linear quantity has a corresponding angular counterpart. The inertia tensor should be viewed as a sort of "rotational mass". The equations of motion in the last line will be used in subsection 8.3 below.

is $\delta m = \rho(\mathbf{r})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$. Here $\mathbf{r}$ is measured from $O$. This effectively replaces $m_I \to \delta m$ and $\mathbf{r}_I \to \mathbf{r}$ in the point particle model, where now $\mathbf{r}$ is a continuous variable that is integrated over. The Riemann integral is, after all, the limit of a sum in this way. The total mass hence becomes

$$M = \iiint_{\mathrm{body}} \rho(\mathbf{r})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{8.27}$$

Similarly, the inertia tensor (8.21) becomes

$$\mathcal{I}_{ij} = \iiint_{\mathrm{body}} \rho(\mathbf{r})\left[(\mathbf{r}\cdot\mathbf{r})\delta_{ij} - r_i r_j\right]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{8.28}$$

Here $\mathbf{r} = (r_1, r_2, r_3) = (x, y, z)$, so that the last equation more explicitly reads

$$\mathcal{I} = \iiint_{\mathrm{body}} \rho(\mathbf{r}) \begin{pmatrix} y^2+z^2 & -xy & -zx \\ -xy & z^2+x^2 & -yz \\ -zx & -yz & x^2+y^2 \end{pmatrix} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\ . \tag{8.29}$$

Note carefully the form of the terms in this matrix.

**Definition** The *moment of inertia about an axis* $\mathbf{n}$ *through* $O$ is $I = \mathbf{n}^{\mathrm{T}}\,\mathcal{I}\,\mathbf{n}$.

In particular, the diagonal entries in (8.29) are the moments of inertia about the three axes. The off-diagonal entries are called the *products of inertia*.

**Example** (Uniform rectangular cuboid): We will only consider *uniform* distributions of mass, in which the density $\rho$ = constant. If we take the cuboid to have side lengths $2a$, $2b$, $2c$ and mass $M$, then $\rho = M/(8abc)$. The centre of mass is the origin of the cuboid, and we take Cartesian axes aligned with the edges. It is then straightforward to see that the products of inertia in this basis are zero; for example

$$\mathcal{I}_{12} = -\frac{M}{8abc}\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} xy\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0\ . \tag{8.30}$$

We next compute

$$\int_{x=-a}^{a}\int_{y=-b}^{b}\int_{z=-c}^{c} \rho\,x^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{M}{8abc}\left[\frac{1}{3}x^3\right]_{-a}^{a} 2b\cdot 2c = \frac{Ma^2}{3}\ . \tag{8.31}$$

The integrals involving $y^2$ and $z^2$ are of course similar, and we deduce that

$$\mathcal{I}^{(G)} \quad = \quad \begin{pmatrix} \frac{1}{3}M(b^2+c^2) & 0 & 0 \\ 0 & \frac{1}{3}M(c^2+a^2) & 0 \\ 0 & 0 & \frac{1}{3}M(a^2+b^2) \end{pmatrix} . \qquad (8.32)$$

■

The inertia tensor (8.32) is diagonal in this last example. Since $\mathcal{I}$ is always a real symmetric matrix, by the Spectral Theorem in Linear Algebra II there is always a change of basis by a (constant) orthogonal matrix $\mathcal{P}$ such that $\mathcal{P}\,\mathcal{I}\,\mathcal{P}^{\mathrm{T}}$ is diagonal.

**Definition** In this latter basis $\mathcal{I} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$, and the eigenvalues $I_i$ of $\mathcal{I}$, $i = 1, 2, 3$, are called the *principal moments of inertia*. The corresponding eigenvectors, with which the axes $\mathbf{e}_i$ are aligned, are called the *principal axes*.

A rigid body thus in general determines its own natural choice of rest frame: the origin is the centre of mass $G$, while the axes are the principal axes. In this frame the inertia tensor about $G$ is diagonal. This is the *natural* choice of rest frame, but it isn't always the most convenient choice.

We may also consider two-dimensional bodies, such as a thin flat disc, or one-dimensional bodies such as a rigid rod. In this case one replaces the density $\rho$ by a *surface density*, or *line density*, respectively, and integrates over the surface or curve, respectively.

**Example** (Thin uniform disc): As a two-dimensional example, consider a thin uniform disc of radius $a$ and mass $M$. Thus the surface density is $\rho = M/(\pi a^2)$, and due to the rotational symmetry the centre of mass must be at the origin of the disc. Taking this to be the origin, with the disc lying in the $(x, y)$ plane at $z = 0$, we may introduce polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ in this plane. We then compute

$$\mathcal{I}_{11} \quad = \quad \iint \rho\, y^2 \,\mathrm{d}x\,\mathrm{d}y \quad = \quad \frac{M}{\pi a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^2\sin^2\theta\, r\,\mathrm{d}r\,\mathrm{d}\theta \quad = \quad \frac{1}{4}Ma^2 . \qquad (8.33)$$

Notice here that the integrand is $\rho(y^2 + z^2) = \rho\, y^2$, as the body is two-dimensional and lies in the plane $z = 0$. By symmetry we must have $\mathcal{I}_{11} = \mathcal{I}_{22}$ (which is easy enough to check explicitly), and we also compute

$$\mathcal{I}_{33} \quad = \quad \iint \rho\,(x^2+y^2)\,\mathrm{d}x\,\mathrm{d}y \quad = \quad \frac{M}{\pi a^2}\cdot 2\pi\int_{r=0}^{a} r^2\, r\,\mathrm{d}r \quad = \quad \frac{1}{2}Ma^2 . \qquad (8.34)$$

Two of the products of inertia ($\mathcal{I}_{13}$ and $\mathcal{I}_{23}$) are obviously zero, because the disc lies in the plane $z = 0$. The less obvious one is

$$\mathcal{I}_{12} \quad = \quad -\iint \rho\, xy\,\mathrm{d}x\,\mathrm{d}y \quad = \quad -\frac{M}{\pi a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\sin\theta\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta \quad = \quad 0 . \qquad (8.35)$$

Thus

$$\mathcal{I}^{(G)} \;=\; \begin{pmatrix} \frac{1}{4}Ma^2 & 0 & 0 \\ 0 & \frac{1}{4}Ma^2 & 0 \\ 0 & 0 & \frac{1}{2}Ma^2 \end{pmatrix} . \tag{8.36}$$

The axes we chose at the start are hence the principal axes, as the inertia tensor is diagonal. ■

**Example** (Uniform rod): As a one-dimensional example, let's consider a *heavy* rod of length $l$, mass $M$, and hence uniform line density $\rho = M/l$. The centre of mass lies in the centre of the rod, but let us instead compute the moment of inertia about an axis $\mathbf{n}$ perpendicular to the rod, passing through one *end* of the rod. We take $\mathbf{r} = (x, 0, 0)$, so that $x \in [0, l]$ parametrizes the distances of points in the rod from one end at $x = 0$. We note that $y = z = 0$, and thus from (8.29) every entry in the inertia tensor $\mathcal{I}_{ij}$ is zero, apart from $\mathcal{I}_{22} = \mathcal{I}_{33} \equiv I$. This is the moment of inertia about any axis $\mathbf{n}$ perpendicular to the rod (for example $\mathbf{n} = \mathbf{e}_2$ or $\mathbf{n} = \mathbf{e}_3$, or more generally any direction $\mathbf{n} = \cos\psi\, \mathbf{e}_2 + \sin\psi\, \mathbf{e}_3$ lying in the $(y, z)$ plane). Using the analogue of (8.29) for a line density we then compute

$$I \;=\; \int_{x=0}^{l} \rho\, x^2\, \mathrm{d}x \;=\; \frac{M}{l} \cdot \left[\frac{1}{3}x^3\right]_0^l \;=\; \frac{1}{3}Ml^2 . \tag{8.37}$$

We shall use this result in the heavy pendulum example in the next subsection. ■

## 8.3 Simple rigid body motion

In this section we study some simple examples of rigid body motion. In general the instantaneous axis of rotation (the direction that $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ points) itself can depend on time: think of throwing a chopping board into the air (the inertia tensor in this case is modelled by the uniform rectangular cuboid example). Here we content ourselves with studying some simpler situations in which the axis of rotation is fixed, so $\mathbf{n} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$ is a time-*independent* vector. The rotation is then described purely by the angular speed $\omega(t) = |\boldsymbol{\omega}(t)|$.

Before we can discuss dynamics, we first need to know the equations of motion. The centre of mass $G$ of the rigid body satisfies Newton's second law in the form (7.6): that is

$$M\ddot{\mathbf{R}}_G \;=\; \dot{\mathbf{P}} \;=\; \mathbf{F}^{\mathrm{ext}} , \tag{8.38}$$

where $\mathbf{F}^{\mathrm{ext}}$ is the total external force acting on the body. The novel part of the motion for a rigid body is of course its rotation. But we have already derived the equation for this too: (7.11) gives

$$\dot{\mathbf{L}}_G \;=\; \boldsymbol{\tau}_G^{\mathrm{ext}} , \tag{8.39}$$

where $\boldsymbol{\tau}_G^{\mathrm{ext}}$ is the total external torque about $G$. Let's see how to use these in practice.

**Example** (Cylinder rolling down an inclined plane): Consider a uniform circular cylinder of length $l$, radius $a$ and mass $M$. The cylinder rolls under gravity, *without slipping*, down a plane inclined at an angle $\varphi$ to the horizontal. Determine the motion of the cylinder.

**Solution**: Since the motion is effectively two-dimensional, we only need to consider the vertical plane through a line of greatest slope of the inclined plane and the centre of mass $G$ of the cylinder. This is shown in Figure 28.

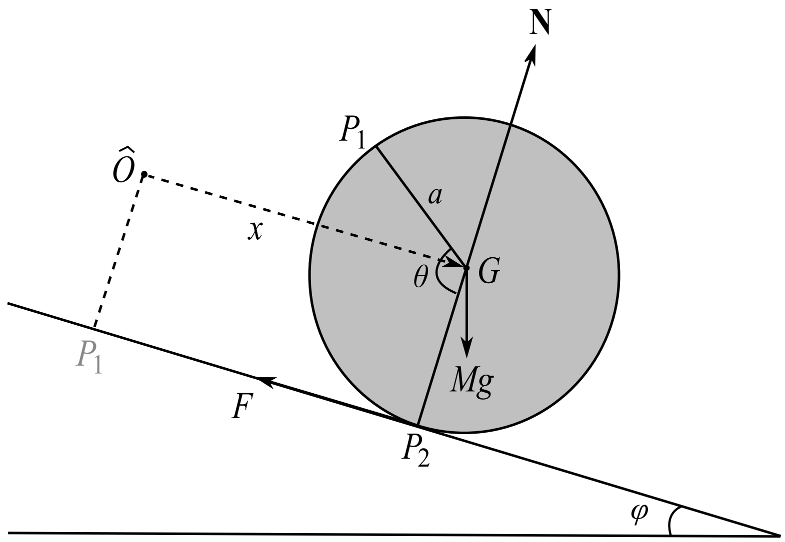



Figure 28: A cross-section through a circular cylinder rolling down a plane inclined at an angle $\varphi$ to the horizontal. The radius of the cylinder is $a$, and the distance travelled down the plane from a fixed origin $\hat{O}$ is $x$. The point of contact with the plane is labelled $P_2$, and a fixed point on the cylinder is labelled $P_1$. The angle between the radius vectors at $P_1$ and $P_2$ is $\theta$, which is the angle through which the cylinder has rolled. A frictional force $F$ acts at $P_2$ up the plane; a normal reaction $\mathbf{N}$ also acts at $P_2$. The gravitational force $Mg$ acts downwards at the centre of mass $G$.

What does it mean to say that the cylinder *rolls without slipping*? By definition, this means that if $x$ is the distanced travelled down the slope and $\theta$ is the angle through which the cylinder has turned, then these are related by

$$x = a\,\theta \,. \tag{8.40}$$

The point here is that $a\,\theta$ is the length of circle segment between points $P_1$ and $P_2$ shown in the Figure. The rotation is purely along the axis of symmetry of the cylinder, which points into the page in Figure 28, through $G$. Taking this to be the $\mathbf{e}_3$ direction, the angular velocity vector is

$$\boldsymbol{\omega} = (0, 0, \dot{\theta}) \,. \tag{8.41}$$

We next need the inertia tensor of the cylinder, about $G$. This is (see Problem Sheet 7)

$$\mathcal{I}^{(G)} = \begin{pmatrix} \frac{1}{12}Ml^2 + \frac{1}{4}Ma^2 & 0 & 0 \\ 0 & \frac{1}{12}Ml^2 + \frac{1}{4}Ma^2 & 0 \\ 0 & 0 & \frac{1}{2}Ma^2 \end{pmatrix} . \tag{8.42}$$

Thus the angular momentum of the cylinder about $G$ is simply

$$\mathbf{L}_G \;=\; (0, 0, I_3\,\dot{\theta})\ , \qquad \text{where} \qquad I_3 \;=\; \frac{1}{2}Ma^2\ . \tag{8.43}$$

Notice here that the axis of rotation is a principal axis, so we only need to know the moment of inertia about this axis for the problem, which is $I_3 = \frac{1}{2}Ma^2$. The rotational form of Newton's second law, in the form (8.39), requires us to find the external torque $\boldsymbol{\tau}_G^{\text{ext}}$ about $G$. There are three forces acting: the normal reaction $\mathbf{N}$, the weight $Mg$, and we have included a frictional force $\mathbf{F}$ of magnitude $F = |\mathbf{F}|$ at the point of contact $P_2$ – see Figure 28. Physically, the friction force is required in order for the cylinder not to slip. The first two of these forces both pass through $G$, and thus have zero moments about $G$: this is immediate for $\mathbf{N}$, while for the weight $Mg$ we are using the Proposition at the end of section 7.2. Thus the only contribution to the torque is from the friction force:

$$\boldsymbol{\tau}_G^{\text{ext}} \;=\; \overrightarrow{GP_2} \wedge \mathbf{F} \;=\; a\,F\,\mathbf{e}_3\ . \tag{8.44}$$

The sign here is easily fixed using the right hand rule. Equation (8.39) thus gives

$$\dot{\mathbf{L}}_G \;=\; (0, 0, I_3\,\ddot{\theta}) \;=\; \boldsymbol{\tau}_G^{\text{ext}} \;=\; (0, 0, a\,F) \qquad \Longrightarrow \qquad I_3\,\ddot{\theta} \;=\; a\,F\ . \tag{8.45}$$

On the other hand, Newton's second law for the centre of mass (8.38) gives

$$M\ddot{x} \;=\; -F + Mg\sin\varphi\ . \tag{8.46}$$

Here the centre of mass motion is in a straight line down the plane, so that $\mathbf{R}_G(t) = (x(t), 0, 0)$. We may eliminate $F$ and $\theta$ in (8.45) using (8.45) and (8.40), giving

$$M\ddot{x} \;=\; -\frac{I_3}{a^2}\ddot{x} + Mg\sin\varphi \tag{8.47}$$

and hence the equation of motion

$$\ddot{x} \;=\; \frac{Ma^2}{I_3 + Ma^2}\,g\sin\varphi \;=\; \frac{2}{3}g\sin\varphi\ . \tag{8.48}$$

It's interesting to compare this result to that for a point particle, sliding down the inclined plane without friction. In this case the equation of motion is $\ddot{x} = g\sin\varphi$. The acceleration of the rolling cylinder is thus reduced by a factor of $2/3$ compared to the point particle. ■

One can equivalently solve the last problem by thinking about energy. For this we need to know the gravitational potential energy of a rigid body:

**Proposition** The total gravitational potential energy of a rigid body in a uniform gravitational field is as if all the mass was located at the centre of mass $G$. That is,

$$V \;=\; Mg\,Z_G\ , \tag{8.49}$$

where $Z_G$ is the $z$ coordinate of the centre of mass $G$.

**Proof**: Here we're of course taking a uniform gravitational field of strength $g$ in the downward $z$ direction. Thinking of the rigid body as made up of masses $\delta m = \rho(\mathbf{r})\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$ at positions $\mathbf{R} = \mathbf{R}_G + \mathbf{r} = (X, Y, Z)$ relative to the origin $\hat{O}$ of an inertial frame, these each have potential energy $\delta m\, g\, Z$. The total potential energy is hence

$$V \;=\; \iiint_{\text{body}} \rho(\mathbf{r})\, g\, Z\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \;=\; Mg\, Z_G\ , \tag{8.50}$$

where the last step follows since by definition (7.2)

$$M\mathbf{R}_G \;=\; \iiint_{\text{body}} \rho(\mathbf{r})\, \mathbf{R}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\ , \tag{8.51}$$

and $\mathbf{R}_G = (X_G, Y_G, Z_G)$. ■

Recall from (8.25) and (8.26) that the kinetic energy is

$$T \;=\; \frac{1}{2}M|\dot{\mathbf{R}}_G|^2 + \frac{1}{2}\sum_{i=1}^{3}\mathcal{I}_{ij}^{(G)}\, \omega_i\, \omega_j\ . \tag{8.52}$$

**Example** (Rolling cylinder again): The cylinder in our example rotates about a fixed axis $\mathbf{e}_3$ with principal moment of inertia $I_3$. Then (8.52) simplifies to

$$T \;=\; \frac{1}{2}M|\mathbf{v}_G|^2 + \frac{1}{2}I_3\, \omega^2 \;=\; \frac{1}{2}M\dot{x}^2 + \frac{1}{2}I_3\, \dot{\theta}^2\ . \tag{8.53}$$

From (8.50) the gravitational potential energy is

$$V \;=\; Mg\, Z_G \;=\; -Mg\, x \sin\varphi\ . \tag{8.54}$$

Thus the total energy is

$$E \;=\; T + V \;=\; \frac{1}{2}\left(M + \frac{I_3}{a^2}\right)\dot{x}^2 - Mg\, x \sin\varphi\ , \tag{8.55}$$

where we have substituted for $\theta$ in terms of $x$ using (8.40). Since there is a frictional force $F$ acting one might be worried that this energy is not conserved. However, the point of contact $P_2$ is always instantaneously at rest, which means that the friction does no work. As usual the normal reaction also does no work, and so energy is indeed conserved. We can see this by taking the time derivative of (8.55)

$$\dot{E} \;=\; \left(M + \frac{I_3}{a^2}\right)\dot{x}\, \ddot{x} - Mg\, \dot{x} \sin\varphi \;=\; \dot{x}\left[\frac{I_3 + Ma^2}{a^2}\ddot{x} - Mg\, \sin\varphi\right]\ . \tag{8.56}$$

Thus energy is conserved if the equation of motion (8.48) holds. ■

**Example** (Heavy pendulum): A *heavy pendulum* consists of a uniform rigid rod of mass $M$ and length $l$, pivoted freely at one end at the origin $O$. The rod swings freely in a vertical plane under gravity. Determine the equation of motion for $\theta$, the angle the rod makes with the vertical.

**Solution**: Notice in this example that we may take the origin $\hat{O}$ of the inertial frame to be the same point as the end of the rod $O$. It's then easier to consider the angular momentum about $O$, rather than about $G$.

The diagram in this case is identical to that for the simple pendulum, Figure 15, except that the mass $M$ is now distributed uniformly along the rod, rather than a point mass $m$ at the end of the rod. We make use of the same polar coordinates (5.5) in the plane of motion, *i.e.* $\mathbf{e}_r = -\cos\theta\,\mathbf{k} + \sin\theta\,\mathbf{i}$, $\mathbf{e}_\theta = \sin\theta\,\mathbf{k} + \cos\theta\,\mathbf{i}$, where the vector $\mathbf{j}$ points into the page in Figure 15. The latter is the axis of rotation of the rod, so we may immediately write the angular velocity vector $\boldsymbol{\omega} = -\dot{\theta}\,\mathbf{j}$. Here the sign is most easily checked using the right hand rule. We calculated the moment of inertia about the axis $\mathbf{j}$ through $O$ in (8.37), giving $I = \frac{1}{3}Ml^2$. Thus the angular momentum is $\mathbf{L}_O = -I\,\dot{\theta}\,\mathbf{j}$.

Notice that we cannot apply (8.39), because we are working about the end of the rod $O = \hat{O}$ rather than $G$. However, we may instead use (7.11) with $P = O = \hat{O}$, which says $\dot{\mathbf{L}}_O = \boldsymbol{\tau}_O^{\rm ext}$. The total external torque here just arises from the weight of the rod, and we may hence use the Proposition at the end of section 7.2. This says the torque is the same as that for a point mass $M$ at the centre of mass $G$, which is half way along the rod:

$$\boldsymbol{\tau}_O^{\rm ext} \;=\; \overrightarrow{OG} \wedge (-Mg\,\mathbf{k}) \;=\; -\frac{l}{2}\mathbf{e}_r \wedge Mg\,\mathbf{k} \;=\; \frac{1}{2}Mgl\sin\theta\,\mathbf{j}\;, \qquad (8.57)$$

where in the last step we have used $\mathbf{e}_r \wedge \mathbf{k} = -\sin\theta\,\mathbf{j}$. Putting everything together, the angular equation of motion reads

$$\dot{\mathbf{L}}_O \;=\; -I\,\ddot{\theta}\,\mathbf{j} \;=\; \frac{1}{2}Mgl\sin\theta\,\mathbf{j} \;=\; \boldsymbol{\tau}_O^{\rm ext}\;. \qquad (8.58)$$

Using $I = \frac{1}{3}Ml^2$ hence gives the equation of motion

$$\ddot{\theta} \;=\; -\frac{3g}{2l}\sin\theta\;. \qquad (8.59)$$

There is an extra factor of 3/2 compared with a simple pendulum of the same mass $M$ and length $l$ – see (5.8). In other words, a heavy pendulum behaves exactly the same as a *simple pendulum* with 2/3 of the length. ■

## 8.4 Newton's laws in a non-inertial frame

Throughout these lectures we've emphasized that Newton's laws (in particular the second law) should always be formulated in an inertial frame. By definition, this is a frame of reference in which Newton's first law holds. On the other hand, we've also mentioned that the Earth is rotating about its axis once per day, and that the Earth accelerates about the Sun on its elliptical (in fact roughly circular, with eccentricity $e_{\rm Earth} \simeq 0.0167$) orbit. A fixed frame relative to the surface of the Earth is then only approximately an inertial frame. What effect does this have, and more generally can we formulate Newton's laws in a general reference frame?

We begin with the same set up as section 8.1: $\hat{\mathcal{S}}$ is a fixed inertial frame with origin $\hat{O}$, and $\mathcal{S}$ is another frame whose origin $O$ is at position vector $\mathbf{x}(t)$, measured from $\hat{O}$. See Figure 25. Suppose that a point particle has position vector $\mathbf{R}$ measured from $\hat{O}$, and $\mathbf{r}$ measured from $O$, as in (8.14). Then

$$\mathbf{R} \;=\; \mathbf{x}+\mathbf{r}\,. \tag{8.60}$$

Recall also from section 8.1 that

**Definition** The *time derivative* of a vector $\mathbf{q}=\mathbf{q}(t)$ in a frame $\mathcal{S}$ is

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}}\mathbf{q} \;=\; \sum_{i=1}^{3}\dot{q}_i\,\mathbf{e}_i\,, \tag{8.61}$$

where $\mathbf{q} = \sum_{i=1}^3 q_i\,\mathbf{e}_i$ and $\{\mathbf{e}_i\}$ is the orthonormal basis for $\mathcal{S}$. That is, we differentiate the *components* of $\mathbf{q}$ in this basis, with respect to time $t$.

The Coriolis formula (8.12) relates the time derivatives of the same vector $\mathbf{q}$ in $\mathcal{S}$ and $\hat{\mathcal{S}}$ as

$$\left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}} \;=\; \left(\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\mathbf{q}\,, \tag{8.62}$$

where $\boldsymbol{\omega}=\boldsymbol{\omega}(t)$ is the angular velocity of $\mathcal{S}$ relative to $\hat{\mathcal{S}}$. By definition then the accelerations $\hat{\mathbf{a}}$ and $\mathbf{a}$ of our particle, as measured in the frames $\hat{\mathcal{S}}$ and $\mathcal{S}$, respectively, are

$$\begin{aligned}\hat{\mathbf{a}} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}\mathbf{R} \;=\; \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}(\mathbf{x}+\mathbf{r}) \;=\; \left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{\hat{\mathcal{S}}} + \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\hat{\mathcal{S}}}\,,\\ \mathbf{a} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\mathcal{S}}\mathbf{r}\,.\end{aligned} \tag{8.63}$$

In order to write down Newton's second law in the frame $\mathcal{S}$ we need the following result:

**Proposition** The accelerations in the two frames are related by

$$\hat{\mathbf{a}} \;=\; \mathbf{a} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}}\wedge\mathbf{r} + 2\boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) + \mathbf{A}\,, \tag{8.64}$$

where we have defined $\mathbf{A} = \left(\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2}\right)_{\hat{\mathcal{S}}}$, which is the acceleration of $O$ relative to $\hat{\mathcal{S}}$.

**Proof**: We compute

$$\begin{aligned}\hat{\mathbf{a}} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}(\mathbf{x}+\mathbf{r}) \;=\; \mathbf{A} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^2_{\hat{\mathcal{S}}}\mathbf{r}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}}\left[\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\mathbf{r}\right] + \mathbf{A}\\ &= \mathbf{a} + \boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}}(\boldsymbol{\omega}\wedge\mathbf{r}) + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) + \mathbf{A}\\ &= \mathbf{a} + \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}}\wedge\mathbf{r} + 2\boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) + \mathbf{A}\,.\end{aligned} \tag{8.65}$$

Here in the third equality we have used the Coriolis formula (8.62) for one of the two time derivatives for $\hat{\mathcal{S}}$. The fourth equality then uses the formula again, with the final step using the product rule for derivatives. For example, you can check from the definition that $\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{\mathcal{S}}(\mathbf{b}\wedge\mathbf{c}) = \left(\frac{\mathrm{d}\mathbf{b}}{\mathrm{d}t}\right)_{\mathcal{S}}\wedge\mathbf{c}+\mathbf{b}\wedge\left(\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}t}\right)_{\mathcal{S}}$, for any two vectors $\mathbf{b}$, $\mathbf{c}$. ■

Notice that using the Coriolis formula (8.62) we have

$$\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\hat{\mathcal{S}}} = \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} + \boldsymbol{\omega}\wedge\boldsymbol{\omega} = \left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}} , \tag{8.66}$$

so that the time derivative of $\boldsymbol{\omega}$ is the same in either frame.

Newton's second law for a particle of mass $m$ in the inertial frame $\hat{\mathcal{S}}$ is

$$m\hat{\mathbf{a}} = \mathbf{F} , \tag{8.67}$$

where $\mathbf{F}$ is the external force acting. Substituting for $\hat{\mathbf{a}}$ in terms of $\mathbf{a}$ using (8.64), we thus have:

**Theorem** Newton's second law in the frame $\mathcal{S}$ is

$$m\mathbf{a} = \mathbf{F} - m\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}}\wedge\mathbf{r} - 2m\,\boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} - m\,\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) - m\mathbf{A} . \tag{8.68}$$

Here the particle's position measured from the origin $O$ of $\mathcal{S}$ is $\mathbf{r}$, $\mathbf{A}$ is the acceleration of $O$, and $\boldsymbol{\omega}$ is the angular velocity of $\mathcal{S}$ (relative to the inertial frame $\hat{\mathcal{S}}$).

The additional terms on the right hand side of (8.68) may be interpreted as "fictitious forces":

$$\begin{aligned}
\mathbf{F}_1 &= -m\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}}\wedge\mathbf{r} , \qquad & \mathbf{F}_2 &= -2m\,\boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} , \\
\mathbf{F}_3 &= -m\,\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) , \qquad & \mathbf{F}_4 &= -m\mathbf{A} .
\end{aligned} \tag{8.69}$$

These may be regarded as corrections to the force in $\mathbf{F}=m\mathbf{a}$ due to the fact that the frame $\mathcal{S}$ is accelerating. The force $\mathbf{F}_1$ is known as the *Euler force*, and arises from the *angular acceleration* of $\mathcal{S}$. The Euler force is hence zero for a frame rotating at constant angular velocity, $\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}\right)_{\mathcal{S}}=\mathbf{0}$. The force $\mathbf{F}_2$ is known as the *Coriolis force*, and is interesting in that it depends on the velocity $\mathbf{v}=\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}}$ of the particle as measured in $\mathcal{S}$. We discuss this force in more detail in section 8.5. The force $\mathbf{F}_3$ is the *centrifugal force*. It lies in a plane through $\mathbf{r}$ and $\boldsymbol{\omega}$, is perpendicular to the axis of rotation $\boldsymbol{\omega}$, and is directed away from the axis. This is the force you experience standing on a roundabout, that seems to throw you outwards. Finally, $\mathbf{F}_4$ is simply due to the acceleration of the origin $O$. For example, this force effectively cancels the Earth's gravitational field in a freely falling frame.

**Corollary** The frame $\mathcal{S}$ is inertial if and only if $\mathbf{A}=\mathbf{0}=\boldsymbol{\omega}$. That is, the origin $O$ is not accelerating, and the basis is not rotating.

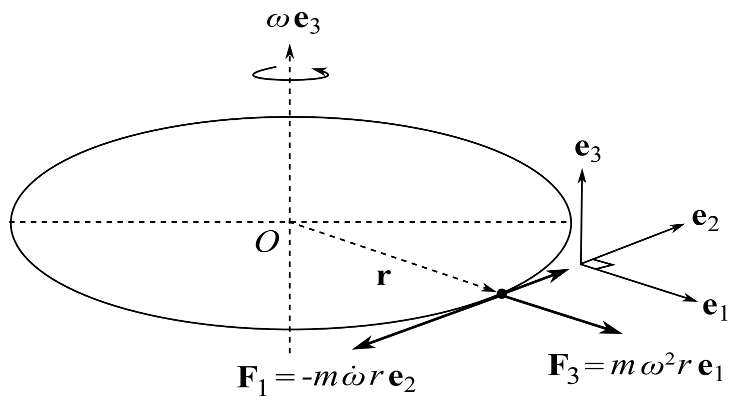



Figure 29: The Euler force $\mathbf{F}_1$ and centrifugal force $\mathbf{F}_3$ in a roundabout frame. Here $O = \hat{O}$, $\mathbf{e}_1$ is a unit vector directed radially outwards, $\mathbf{e}_2$ is a unit vector orthogonal to this in the horizontal plane of the roundabout, and $\mathbf{e}_3$ is a unit vector in the direction of the axis of rotation. The position vector of particle of mass $m$ is $\mathbf{R} = \mathbf{r} = r\,\mathbf{e}_1$. The Euler force is then $\mathbf{F}_1 = -m\dot{\omega}\,\mathbf{e}_3 \wedge \mathbf{r} = -m\dot{\omega}r\,\mathbf{e}_2$ while the centrifugal force is $-m\omega\,\mathbf{e}_3 \wedge (\omega\,\mathbf{e}_3 \wedge \mathbf{r}) = m\omega^2 r\,\mathbf{e}_1$.

* **Proof**: First note that the frame $\mathcal{S}$ being inertial means that any particle with no force acting ($\mathbf{F} = \mathbf{0}$) moves at constant velocity in the frame $\mathcal{S}$. If $\mathbf{A} = \mathbf{0} = \boldsymbol{\omega}$ then (8.68) with $\mathbf{F} = \mathbf{0}$ immediately gives $\mathbf{a} = \mathbf{0}$, and hence the particle moves with constant velocity in $\mathcal{S}$. Conversely, suppose that $\mathbf{F} = \mathbf{0}$ and a particle moves with constant velocity $\mathbf{r}(t) = \mathbf{u}\,t + \mathbf{r}_0$ in $\mathcal{S}$. Here $\mathbf{u}$ and $\mathbf{r}_0$ are *arbitrary* constant vectors in $\mathcal{S}$ (effectively integration constants from integrating $\mathbf{a} = \mathbf{0}$). First setting $\mathbf{u} = \mathbf{r}_0 = \mathbf{0}$ (so the particle is fixed at the origin of $\mathcal{S}$), we immediately deduce from substituting $\mathbf{r} \equiv \mathbf{0}$ into (8.68) that $\mathbf{A} = \mathbf{0}$. Next, for *fixed* time $t = t_0$ we may set $\mathbf{r}_0 = -\mathbf{u}\,t_0$ (so the particle is at the origin of $\mathcal{S}$ at time $t_0$), and again substitute for $\mathbf{r}(t) = \mathbf{u}\,t + \mathbf{r}_0$ into (8.68). Evaluated at time $t = t_0$, the only term that survives is the Coriolis term $-2m\,\boldsymbol{\omega}(t_0) \wedge \mathbf{u}$, which must be zero for all $\mathbf{u}$. But this implies that $\boldsymbol{\omega}(t_0) = \mathbf{0}$, and since $t_0$ was arbitrary hence $\boldsymbol{\omega} \equiv \mathbf{0}$. ■

Newton's second law (8.68) may be used to solve dynamics problems in rotating frames. In principle this is straightforward, but in practice one needs to be careful! In the two examples that follow the origin $O$ of the rotating frame $\mathcal{S}$ may be taken to coincide with $\hat{O}$, so that $\mathbf{x} = \mathbf{0}$ and the position vectors in the two frames are equal $\mathbf{R} = \mathbf{r}$.

**Example** (Bead on a rotating, smooth, straight horizontal wire): Consider a bead (point particle) that slides on a frictionless straight horizontal wire. The wire is fixed at $O = \hat{O}$, and rotates in a horizontal plane at constant angular velocity $\omega$. Determine the motion of the bead.

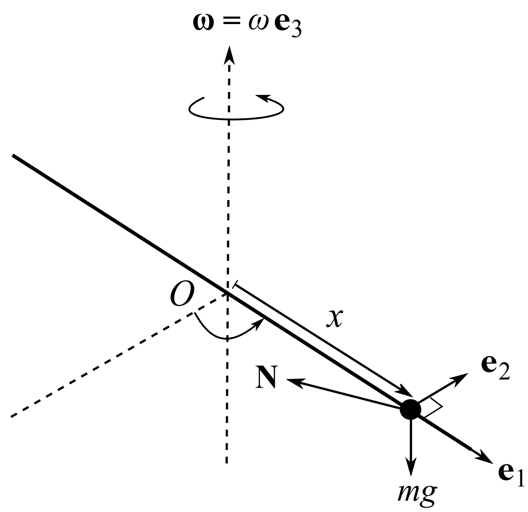



Figure 30: The bead on the rotating horizontal wire. The forces acting on the bead are $-mg\,\mathbf{e}_3$ and the normal reaction $\mathbf{N}$ perpendicular to the wire.

**Solution**: We choose the rotating basis $\{\mathbf{e}_i\}$ for $\mathcal{S}$ as follows: $\mathbf{e}_1$ is a unit vector pointing along the wire, $\mathbf{e}_2$ is a unit horizontal vector normal to the wire, and $\mathbf{e}_3$ is a unit vector vertically. The position of the bead is hence $\mathbf{r} = \mathbf{R} = x\,\mathbf{e}_1$, while the angular velocity of the frame is $\boldsymbol{\omega} = \omega\,\mathbf{e}_3$. Denoting the normal reaction of the wire on the bead by $\mathbf{N}$, the total "real force" acting on the bead is

$$\mathbf{F} \;=\; \mathbf{N} - mg\,\mathbf{e}_3\;. \tag{8.70}$$

However, the frame is rotating, so we must use Newton's second law in the form (8.68). Since $\boldsymbol{\omega}$ is constant and $\mathbf{A} = \mathbf{0}$ the second and last terms on the right hand side of (8.68) are zero, and we have

$$m\,\ddot{x}\,\mathbf{e}_1 \;=\; \mathbf{F} - 2m\omega\,\dot{x}\,\mathbf{e}_2 + m\omega^2 x\,\mathbf{e}_1\;. \tag{8.71}$$

Here we've used $\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} = \dot{x}\,\mathbf{e}_1$, so that the Coriolis force is

$$\mathbf{F}_2 \;=\; -2m\,\boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} \;=\; -2m\omega\,\mathbf{e}_3\wedge\dot{x}\,\mathbf{e}_1 \;=\; -2m\omega\,\dot{x}\,\mathbf{e}_2\;, \tag{8.72}$$

while the centrifugal force is

$$\mathbf{F}_3 \;=\; -m\,\boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{r}) \;=\; -m\omega^2\,\mathbf{e}_3\wedge(\mathbf{e}_3\wedge x\,\mathbf{e}_1) \;=\; m\omega^2\,x\,\mathbf{e}_1\;. \tag{8.73}$$

As in section 5.1, the wire being *smooth* means that the normal reaction $\mathbf{N}$ has no component along the wire, $\mathbf{N} \cdot \mathbf{e}_1 = 0$. Thus taking the dot product of (8.71) with $\mathbf{e}_1$ gives simply

$$m\ddot{x} \;=\; m\omega^2 x \; . \tag{8.74}$$

The general solution is

$$x(t) \;= A\,\mathrm{e}^{\omega t} + B\,\mathrm{e}^{-\omega t} \; . \tag{8.75}$$

For example, if the bead starts at a distance $x = a$ from $O$ with $\dot{x} = 0$ at time $t = 0$, then

$$x(t) \;=\; \frac{a}{2}(\mathrm{e}^{\omega t} + \mathrm{e}^{-\omega t}) \;=\; a\,\cosh\omega t \; . \tag{8.76}$$

The bead hence flings outwards along the wire, with $x(t)$ growing exponentially with $t$. ■

**Example** (Bead on a rotating smooth hoop): A circular hoop of radius $a$ rotates at constant angular velocity $\omega$ about a vertical diameter. A bead slides smoothy on the hoop and has a position vector which makes an angle $\varphi$ with the vertical, as in Figure 31. Show that the equation of motion is

$$\ddot{\varphi} + \left(\frac{g}{a} - \omega^2\cos\varphi\right)\sin\varphi \;=\; 0 \; . \tag{8.77}$$

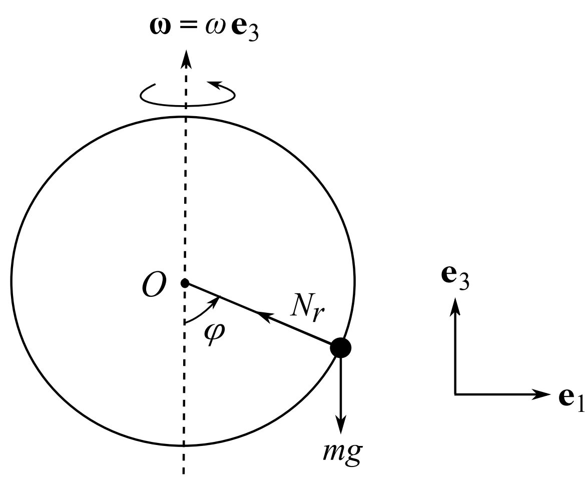



Figure 31: The bead on the rotating hoop. Here the figure shows the hoop at the instant at which it passes through the plane of the page. The component $N_2$ of the normal reaction $\mathbf{N}$ of the hoop on the bead points into the page at this instant, which is the $\mathbf{e}_2$ direction.

**Solution**: We take the origins $O = \hat{O}$ to be the centre of the hoop, and the frame $\mathcal{S}$ to be the rest frame of the hoop. In particular we take $\mathbf{e}_1$ to be a horizontal unit vector and $\mathbf{e}_3$ to be a vertical unit vector, which define the (rotating) plane of the hoop. We may then parametrize the position of the bead as

$$\mathbf{r} \;=\; \mathbf{R} \;=\; a\,\sin\varphi\,\mathbf{e}_1 - a\,\cos\varphi\,\mathbf{e}_3 \; . \tag{8.78}$$

We then compute the velocity and acceleration of the bead with respect to the rotating frame:

$$\begin{aligned} \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}} &= a\,\dot{\varphi}\,\cos\varphi\,\mathbf{e}_1 + a\,\dot{\varphi}\,\sin\varphi\,\mathbf{e}_3 \ , \\ \mathbf{a} \ = \ \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\mathcal{S}} &= a(\ddot{\varphi}\,\cos\varphi - \dot{\varphi}^2\sin\varphi)\,\mathbf{e}_1 + a(\ddot{\varphi}\,\sin\varphi + \dot{\varphi}^2\cos\varphi)\,\mathbf{e}_3 \ . \end{aligned} \tag{8.79}$$

Denoting $\mathbf{N}$ the normal reaction of the hoop on the bead, the force on the bead is again given by (8.70). The angular velocity is $\boldsymbol{\omega} = \omega\,\mathbf{e}_3$, and Newton's second law (8.68) hence reads

$$\begin{aligned} m\mathbf{a} &= ma(\ddot{\varphi}\,\cos\varphi - \dot{\varphi}^2\sin\varphi)\,\mathbf{e}_1 + ma(\ddot{\varphi}\,\sin\varphi + \dot{\varphi}^2\cos\varphi)\,\mathbf{e}_3 \qquad (8.80) \\ &= \mathbf{F} - 2m\omega\,\mathbf{e}_3 \wedge (a\,\dot{\varphi}\,\cos\varphi\,\mathbf{e}_1 + a\,\dot{\varphi}\,\sin\varphi\,\mathbf{e}_3) - m\omega\,\mathbf{e}_3 \wedge [\omega\,\mathbf{e}_3 \wedge (a\,\sin\varphi\,\mathbf{e}_1 - a\,\cos\varphi\,\mathbf{e}_3)] \ . \end{aligned}$$

Once again notice that we only have the Coriolis and centrifugal terms as "fictitious forces" on the right hand side. Computing the wedge products in (8.80) simplifies the latter to

$$\begin{aligned} ma(\ddot{\varphi}\,\cos\varphi - \dot{\varphi}^2\sin\varphi)\,\mathbf{e}_1 + ma(\ddot{\varphi}\,\sin\varphi + \dot{\varphi}^2\cos\varphi)\,\mathbf{e}_3 &= \mathbf{N} - mg\,\mathbf{e}_3 - 2m\omega\,a\,\dot{\varphi}\,\cos\varphi\,\mathbf{e}_2 \\ &\quad + m\omega^2\,a\,\sin\varphi\,\mathbf{e}_1 \ . \end{aligned} \tag{8.81}$$

The normal reaction $\mathbf{N}$ has a radial component $N_r$ (see Figure 31) and a component $N_2$ into the page. Thus

$$\mathbf{N} \ = \ N_r(-\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_3) + N_2\,\mathbf{e}_2 \ . \tag{8.82}$$

We could now equate components of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ in (8.81) to give three scalar equations for the three unknowns $\varphi$, $N_r$ and $N_2$. Eliminating $N_r$ and $N_2$ would then give an equation for $\varphi$. However, a quicker method is to note that $\mathbf{N}$ is orthogonal to the tangent of the circular hoop, so that taking the dot product of (8.81) with this tangent vector will immediately eliminate $\mathbf{N}$. The tangent vector is

$$\mathbf{t} \ = \ \cos\varphi\,\mathbf{e}_1 + \sin\varphi\,\mathbf{e}_3 \ , \tag{8.83}$$

and taking the dot product with (8.81) gives (using $\cos^2\varphi + \sin^2\varphi = 1$)

$$ma\,\ddot{\varphi} \ = \ -mg\,\sin\varphi + m\omega^2\,a\,\sin\varphi\,\cos\varphi \ . \tag{8.84}$$

Dividing through by $ma$ then gives the required equation of motion (8.77). ■

## 8.5 * The Coriolis force

We are unlikely to have time to discuss the content of this section in lectures: you may treat it as starred.

You might have noticed in these last two examples that the only fictitious force that entered the equations of motion (8.74), (8.77) was the centrifugal force $\mathbf{F}_3$ in (8.69). The Coriolis force $\mathbf{F}_2$ instead determined the normal reaction. For example, in the last example $N_2 = 2m\omega\,a\,\dot{\varphi}\,\cos\varphi$, which is precisely due to the Coriolis force (look at the $\mathbf{e}_2$ component of (8.81)). In general the Coriolis force is

$$\mathbf{F}_2 \;=\; \mathbf{F}_{\text{Coriolis}} \;=\; -2m\,\boldsymbol{\omega}\wedge\mathbf{v}\ , \tag{8.85}$$

where $\mathbf{v} = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}}$ is the velocity of the particle as measured in the rotating frame $\mathcal{S}$. It is this velocity dependence that leads to some of the more peculiar features of the dynamics, compared to the other fictitious forces. In fact *mathematically* the Coriolis force is equivalent to the magnetic component of the Lorentz force law (2.26), with the angular velocity playing the role of the magnetic field. The dynamics generated by the two forces is hence similar. The effects of both Coriolis and centrifugal forces in a frame fixed to the rotating surface of the Earth are both rather small in everyday life (the Euler force $\mathbf{F}_1$ being even more negligible, as the rate of rotation of the Earth is very nearly constant at $\omega = 2\pi$ radians per day). In this section we consider a famous set up that demonstrates the dynamics driven by the Coriolis force on Earth: *Foucault's pendulum.*

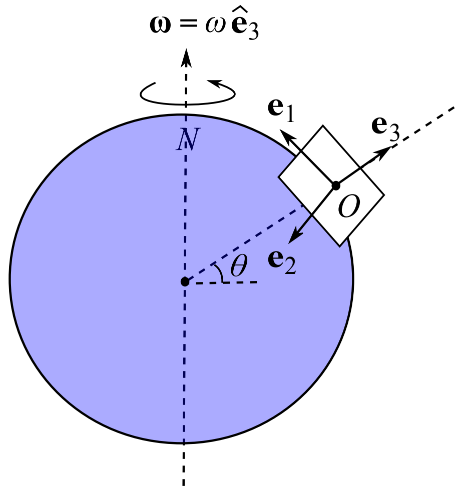



Figure 32: A frame $\mathcal{S}$ fixed to the surface of the rotating Earth. The angular velocity $\omega = 2\pi$ radians per day, or $\omega \simeq 7\times 10^{-5}\ \mathrm{s}^{-1}$. The latitude of the origin $O$ of $\mathcal{S}$ is $\theta$.

What does it mean to have a reference frame $\mathcal{S}$ fixed to the surface of the rotating Earth? This is shown in Figure 32. We take $\mathbf{e}_1$ to be a unit vector pointing North, and $\mathbf{e}_2$ a unit vector pointing West. $\mathbf{e}_3$ is a radial vector from the centre of the Earth pointing outwards, so that on the surface of the Earth this is a unit vector pointing up. On the other hand, the Earth rotates about its axis $\hat{\mathbf{e}}_3$, so that the angular velocity is $\boldsymbol{\omega} = \omega\,\hat{\mathbf{e}}_3$. If we are at a constant latitude $\theta$, then the relation between these vectors is

$$\hat{\mathbf{e}}_3 \;=\; \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_3\ . \tag{8.86}$$

In this case the origin $O$ is moving in a circle about the Earth's axis, and is thus accelerating with respect to the centre of mass of the Earth. Taking the centre of mass of the Earth to be the origin $\hat{O}$ of an inertial frame (hence ignoring its motion about the Sun), the acceleration $\mathbf{A}$ in Newton's second law (8.68) is a *centripetal acceleration* for this circular motion. From (4.8) this has magnitude $|\mathbf{A}| = d\,\omega^2$, where $d$ is the distance to the axis. This is hence largest at the equator, where $d = R_E \simeq 6\times 10^6$ m. Using $\omega \simeq 7\times 10^{-5}\,\mathrm{s}^{-1}$ we compute $|\mathbf{A}|_{\max} \simeq 0.03\ \mathrm{m\,s^{-2}}$. This is *very* small compared to $g \simeq 10\ \mathrm{m\,s^{-2}}$, but indeed the *effective* value of $g$ at the equator is slightly smaller than that at the poles due to this effect.

Now consider a pendulum in our rotating frame $\mathcal{S}$. We take the origin $O$ to be at a distance $l$ directly below the pivot (unlike for our previous discussions of pendula), so that when hanging vertically the mass $m$ sits at the origin. We denote the position vector of the mass as $\mathbf{r} = (x, y, z)$ in the basis $\{\mathbf{e}_i\}$. See Figure 33. The light rod constraints these coordinates via $x^2+y^2+(l-z)^2 = l^2$.

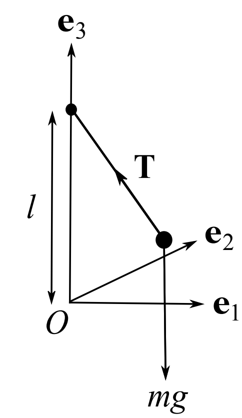



Figure 33: Foucault's pendulum. The position of the mass $m$ is $\mathbf{r} = (x, y, z) = x\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3$, with coordinates constrained via $x^2 + y^2 + (l-z)^2 = l^2$.

In this problem we're *only* interested in the effect of the Coriolis force on the motion of the pendulum, which turns out to be the most important term on the right hand side of (8.68). We may thus write the equation of motion (8.68) as

$$m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{\mathcal{S}} \simeq \mathbf{T} - mg\,\mathbf{e}_3 - 2m\,\boldsymbol{\omega}\wedge\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{\mathcal{S}}\ , \tag{8.87}$$

where $\mathbf{T}$ is the tension in the rod. Since we'll now only be computing time derivatives in the frame $\mathcal{S}$, we'll write this more succinctly as (writing also "$=$" rather than "$\simeq$")

$$m\ddot{\mathbf{r}} = \mathbf{T} - mg\,\mathbf{e}_3 - 2m\,\boldsymbol{\omega}\wedge\dot{\mathbf{r}}\ , \tag{8.88}$$

where $\dot{\mathbf{r}} = (\dot{x}, \dot{y}, \dot{z})$, $\ddot{\mathbf{r}} = (\ddot{x}, \ddot{y}, \ddot{z})$. The tension $\mathbf{T}$ has magnitude $\mathbb{T} = |\mathbf{T}|$, and the geometry of Figure 33 implies this is hence

$$\mathbf{T} = \mathbb{T}\left(-\frac{x}{l}\mathbf{e}_1 - \frac{y}{l}\mathbf{e}_2 + \frac{l-z}{l}\mathbf{e}_3\right)\ . \tag{8.89}$$

Using (8.86) the angular velocity is $\boldsymbol{\omega} = \omega\cos\theta\,\mathbf{e}_1 + \omega\sin\theta\,\mathbf{e}_3 = (\omega\cos\theta, 0, \omega\sin\theta)$, and computing the wedge product $\boldsymbol{\omega}\wedge\dot{\mathbf{r}}$ the equation of motion (8.88) hence gives the following coupled ODEs

$$\begin{aligned} m\ddot{x} &= -\frac{x}{l}\mathbb{T} + 2m\omega\,\dot{y}\,\sin\theta\ , \\ m\ddot{y} &= -\frac{y}{l}\mathbb{T} + 2m\omega(\dot{z}\,\cos\theta - \dot{x}\,\sin\theta)\ , \\ m\ddot{z} &= \frac{l-z}{l}\mathbb{T} - mg - 2m\omega\,\dot{y}\,\cos\theta\ . \end{aligned} \tag{8.90}$$

This is a complicated system, but let's look at the equations for a very long pendulum, making small oscillations. This means that the dimensionless quantities $x/l$ and $y/l$ are both small. On the other hand, the constraint equation implies that (for $z<l$)

$$\frac{z}{l} = 1-\sqrt{1-\frac{x^2}{l^2}-\frac{y^2}{l^2}} \simeq \frac{x^2}{2l^2}+\frac{y^2}{2l^2}+\cdots\ . \tag{8.91}$$

Thus $z/l$ is *second order* in $x/l$, $y/l$, and we can hence take $z/l\simeq 0$ in this approximation. The last equation in (8.90) then gives the tension as

$$\mathbb{T} \simeq mg + 2m\omega\,\dot{y}\,\cos\theta \simeq mg\ . \tag{8.92}$$

Here the second approximation follows from the fact that $\omega\simeq 7\times 10^{-5}\,\mathrm{s}^{-1}$ while $g\simeq 10\,\mathrm{m\,s}^{-2}$: the second term in $\mathbb{T}$ in (8.92) is the same order of magnitude as the first term if $\dot{y}\simeq 300,000$ miles per hour! As in the example of a point charge moving in a constant magnetic field in section 2.4, it is next useful to introduce the complex coordinate $\zeta = x+\mathrm{i}y$. Using the approximations we've made, the first two equations in (8.90) then become the real and imaginary parts of

$$\ddot{\zeta} \simeq -\frac{g}{l}\zeta - 2\omega\sin\theta\,\mathrm{i}\,\dot{\zeta}\ . \tag{8.93}$$

Compare this with (2.33) for a charged particle moving in a magnetic field. Substituting the ansatz $\zeta=\mathrm{e}^{pt}$ into (8.93) gives the quadratic equation

$$p^2 + 2\mathrm{i}\,\omega\sin\theta\,p + \frac{g}{l} = 0\ , \tag{8.94}$$

with roots

$$\begin{aligned} p &= -\mathrm{i}\,\omega\sin\theta \pm \sqrt{-\omega^2\sin^2\theta - \frac{g}{l}} = -\mathrm{i}\,\omega\sin\theta \pm \mathrm{i}\sqrt{\frac{g}{l}}\left(1+\frac{\omega^2}{g/l}\sin^2\theta\right)^{1/2} \\ &\simeq -\mathrm{i}\left(\omega\sin\theta\pm\sqrt{\frac{g}{l}}\right)\ . \end{aligned} \tag{8.95}$$

Again, the last approximation follows since $\omega^2$ is extremely small compared with $g/l$, for terrestrial lengths $l$. The solutions may hence be written as

$$\zeta(t) = x(t)+\mathrm{i}\,y(t) \simeq \mathrm{e}^{-\mathrm{i}\omega\sin\theta\,t}\left(C\,\cos\,\omega_0\,t + D\,\sin\,\omega_0\,t\right)\ , \tag{8.96}$$

where $C$ and $D$ are complex integration constants, and $\omega_0=\sqrt{g/l}$ is the usual frequency of small oscillations for the simple pendulum.

The term in brackets on the right hand side of (8.96) in general traces out an ellipse in the $(x, y)$ plane. Our usual simple pendulum confined to the $(x, z)$ plane has $C$ and $D$ real, for which this ellipse degenerates to a line. The phase $\mathrm{e}^{-\mathrm{i}\omega \sin\theta\, t}$ causes the ellipse to rotate in the $(x, y)$ plane. In the Northern hemisphere with $\theta > 0$ this rotation is clockwise (viewed from above), while in the Southern hemisphere with $\theta < 0$ the rotation is anticlockwise. The period of the rotaton is $T = 24/|\sin\theta\,|$ hours, which is minimized at the North and South poles, $\theta = \pm\frac{\pi}{2}$.

Foucault built his original pendulum in 1851, in the Panthéon in Paris. It consisted of a 28 kg metal bob with a 67 m long wire, suspended from the top of the dome. An exact replica has been permanently swinging in the Panthéon since 1995 (apart from quite recently when repair work was carried out). Paris has a latitude of $\theta \simeq 48°$, for which we calculate the period $T \simeq 32$ hours. Said differently, in a single day the pendulum motion has rotated through 270°. Thus if the simple pendulum starts swinging North–South (*i.e.* in a vertical plane), then at the same time the following day it will be swinging East–West. This beautifully matches the (approximate) solution we have found in this subsection.

The Coriolis force also plays an important role in the weather, for example being responsible for the circulation of air around an area of low pressure, which is hence in the opposite directions in the Northern and Southern hemispheres. That's why the direction of spin of a hurricane depends on whether it formed in the Northern or Southern hemisphere, and why hurricanes don't form at all near the equator at $\theta = 0$. ■