Dynamics: Problem Sheet 3 (of 8)

Energy and equilibria.

1. A bead of mass m is attached to the end of a straight spring, where the spring has natural length a and spring constant k. The other end of the spring is fixed at the origin O. The bead and spring hang directly below O, with the spring lying along the vertical line through O.

   (a) Explain why a potential energy function for the bead is

   $$V(x) = \frac{1}{2}k(x-a)^2 - mgx ,$$

   where x is the distance of the bead beneath O. Find the equilibrium position of the bead.

   (b) The bead is released from rest, with the spring at its natural length. Using conservation of energy, find the position of the bead when it is next at rest.

2. (a) A particle moves on the x axis under the influence of a force $F_a(x)$ that is inversely proportional to the square of the distance of the particle from the point $x = a > 0$. Given that this force is also repulsive, show that a potential energy function is

   $$V_a(x) = \frac{\kappa}{|x-a|} ,$$

   where the constant $\kappa > 0$.

   (b) Suppose now that the particle experiences *two* such repulsive forces, one from $x = a$ and the other from $x = -a$, so that the total potential energy function is

   $$V(x) = V_a(x) + V_{-a}(x) .$$

   Show that the origin $x = 0$ is a stable equilibrium, with the (angular) frequency of small oscillations given by $\omega = 2\sqrt{\kappa/ma^3}$.

3. Consider a unit mass ($m = 1$) particle moving in the potential

   $$V(x) = -\frac{2}{3}x^3 - 2x^2 .$$

   (a) Show that $x = -2$ is a stable equilibrium, while $x = 0$ is an unstable equilibrium.

   (b) By sketching the potential argue that there are bounded solutions for energies $E = \frac{1}{2}\dot{x}^2 + V$ in the range $-\frac{8}{3} \leq E < 0$.

   (c) Verify that $x(t) = -3\,\mathrm{sech}^2(t - t_0)$ is a solution with energy $E = 0$. Does the particle ever reach $x = 0$?

   (d) [*Optional*: Compare the behaviour of the solution in part (c), near to $x = 0$, with the general linearized solution about the unstable equilibrium point $x = 0$.]

4. Consider the following differential equation for $\varphi(t)$:

   $$\ddot{\varphi} + \left(\frac{g}{a} - \omega^2 \cos\varphi\right)\sin\varphi = 0 \qquad (*) .$$

   Here φ is a periodic variable, with period 2π, while g, a and ω are all positive constants.

(a) Find the equilibrium positions for φ (*i.e.* find the *constant* solutions to $(*)$), and determine their stability as a function of the parameter $s = g/a\omega^2$.

(b) Sketch the *stable* equilibrium values of φ as a function of $s > 0$.

[In section 8.4 of the lecture notes we will show that equation $(*)$ describes the dynamics of a bead on a hoop of radius a, where the hoop rotates with constant angular velocity ω about a vertical diameter, and the bead makes an angle φ with the vertical. But you do not need to know any of this in order to answer the question!]

Please send comments and corrections to `sparks@maths.ox.ac.uk`.