Dynamics: Problem Sheet 6 (of 8)

Central conservative forces, conservation of angular momentum, Kepler problem.

1. Suppose that a particle moves in response to a central force per unit mass $-f(r)\,\mathbf{e}_r$, where

$$f(r) \;=\; \frac{\alpha}{r^2} + \frac{\beta^2}{r^3} \;.$$

Here r denotes distance to the origin and α, β are constants. Initially the particle is at $r = \beta^2/3\alpha$, $\theta = 0$ and is moving with speed $4\alpha/\beta$ in a direction making an angle of $\pi/3$ with the radius vector pointing towards the origin.

Starting from Newton's second law show that, if $u = 1/r$, then

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \frac{u}{4} \;=\; \frac{3\alpha}{4\beta^2} \;,$$

with

$$u \;=\; \frac{3\alpha}{\beta^2} \;, \qquad \frac{\mathrm{d}u}{\mathrm{d}\theta} \;=\; \frac{\alpha\sqrt{3}}{\beta^2} \qquad \text{when } \theta \;=\; 0 \;.$$

Hence show that the solution is

$$\frac{1}{r} \;=\; \frac{3\alpha}{\beta^2}\left(\frac{2}{\sqrt{3}}\sin\frac{\theta}{2} + 1\right) \;.$$

Sketch the orbit.

2. A particle is dropped from the top of a tower on the Earth's equator. As a result of the Earth's rotation, does it land slightly to the East, or slightly to the West of the tower?

3. A charged particle of charge q and mass m moves in the electric field of a fixed point charge Q, with $q\,Q > 0$.

   (a) Show that the conserved energy of the charge q is

$$E \;=\; \frac{1}{2}m\dot{r}^2 + \frac{mh^2}{2r^2} + \frac{q\,Q}{4\pi\epsilon_0 r} \;,$$

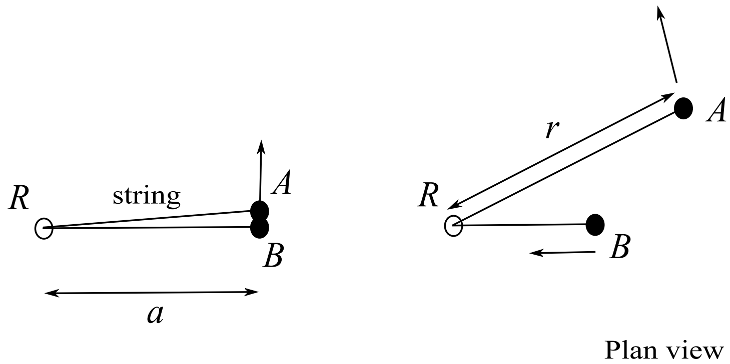
   where r is the distance between the two charges, and h is a constant.

   (b) Initially the charge q is approaching the fixed charge Q with speed v (at a large distance), along a path which, if continued in a straight line, would pass Q at a distance b. Explain why conservation of angular momentum implies $h = vb$, and hence show that the actual distance of closest approach is

$$r_* \;=\; a + \sqrt{a^2 + b^2} \;, \qquad \text{where} \qquad a \;=\; \frac{q\,Q}{4\pi\epsilon_0 m v^2} \;.$$

   [The distance b is called the *impact parameter*.]

4. Two particles A, B of mass m_1, m_2, respectively, lie together on a smooth horizontal table.



They are connected by a light inextensible string of length $2a$ which passes through a light ring R fixed in the table at a distance a from the particles. The ring is smooth and can rotate freely. The particle A is given an initial velocity perpendicular to the string in the plane of the table.

Show that if $u = 1/r$, where r is the distance of A from R, then

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \frac{m_1}{m_1 + m_2} u \quad = \quad 0 \ ,$$

where θ is the angle ARB. Hence find the equation of the path taken by A (up until the moment B reaches R).

[*Hint*: The tension in the string provides a central force for both particles.]

Please send comments and corrections to `sparks@maths.ox.ac.uk`.