Dynamics: Problem Sheet 8 (of 8)

Rigid body motion, Newton's laws in non-inertial frames.

1. (a) Consider a rigid body that is rotating about a general point O that is fixed both in the body *and* fixed in an inertial frame. Starting from the point particle model of a rigid body, show that its kinetic energy is

$$T \equiv \sum_{I=1}^{N} \frac{1}{2} m_I |\dot{\mathbf{r}}_I|^2 = \frac{1}{2} \sum_{i,j=1}^{3} \mathcal{I}_{ij}^{(O)} \omega_i \omega_j ,$$

where $\mathcal{I}^{(O)}$ is the inertia tensor of the body about O, and $\boldsymbol{\omega}$ is its angular velocity. [*Hint*: You might find the vector identity $|\mathbf{a} \wedge \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$ helpful.]

   (b) Using the result in part (a), hence show that the kinetic energy of the heavy pendulum, considered at the end of section 8.3 of the lecture notes, is

$$T = \frac{1}{6} M l^2 \dot{\theta}^2 .$$

Here recall that θ is the angle the pendulum makes with the downward vertical, and the pendulum has length l and mass M.

   (c) Given that the potential energy is $V = MgZ_G$, where Z_G is the height of the centre of mass of the pendulum, hence write down the total energy $E = T + V$. Show that conservation of E is implied by the equation of motion (8.59) derived in the lecture notes.

2. A smooth straight wire rotates with constant angular speed ω about the vertical axis through a fixed point O on the wire, and the angle between the wire and the upward vertical is constant and equal to α, where $0 < \alpha < \pi/2$. A bead of mass m is free to slide on the wire.

   (a) Starting from the general form of Newton's second law in a rotating frame, show that

$$m \left(\frac{\mathrm{d}^2 \mathbf{r}}{\mathrm{d}t^2} \right)_{\mathcal{S}} = \mathbf{N} - mg\,\mathbf{k} - 2m\omega\,\mathbf{k} \wedge \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \right)_{\mathcal{S}} - m\omega^2\,\mathbf{k} \wedge (\mathbf{k} \wedge \mathbf{r}) ,$$

where $\mathbf{r}(t)$ is the position of the bead in the frame $\mathcal{S}$ that rotates with the wire, $\mathbf{k}$ is a unit vector pointing vertically, and $\mathbf{N}$ is the normal reaction of the wire.

   (b) Hence show that $z(t)$, the height of the bead above O, satisfies the equation

$$\ddot{z} - (\omega^2 \sin^2 \alpha) z = -g \cos^2 \alpha .$$

Show that an equilibrium point for the bead exists, and determine its stability.

3. A bead P of mass m slides on a smooth circular wire of radius a and centre C. The wire lies in a horizontal plane and is forced to rotate at a constant angular speed ω about the vertical axis through a fixed point O in the plane of the wire. The distance from O to C is constant and equal to b.

   Let θ be the angle that the line joining the centre C to the bead makes with the diameter through O. (See the figure on the next page.)

(a) Show that the position of the bead may be written as

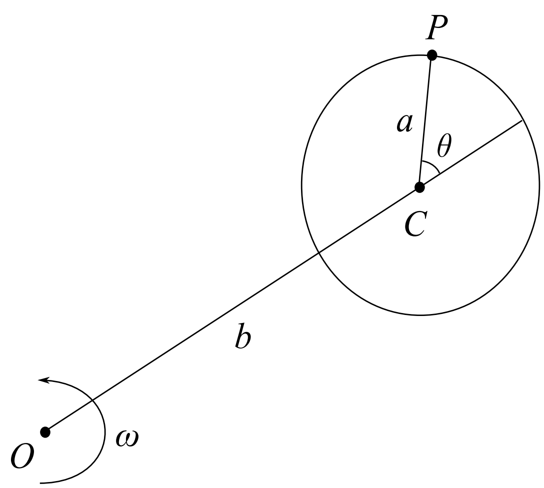
$$\mathbf{r} \quad = \quad (b + a \cos\theta)\, \mathbf{e}_1 + a \sin\theta\, \mathbf{e}_2 \ ,$$

where $\mathbf{e}_i$, $i = 1, 2, 3$, are an orthonormal basis for a frame that rotates with the wire.

(b) Using Newton's second law in this rotating frame, hence show that

$$\ddot{\theta} + \frac{b}{a}\omega^2 \sin\theta \quad = \quad 0 \ .$$

(c) Show that the bead can remain in equilibrium relative to the wire at two points. Decide whether these positions of equilibrium are stable or unstable.



---

4. (*Definitely optional: for those that read section 8.5*)

Consider (again) dropping a particle from the top of a tower, of height h above the ground, at time $t = 0$. We use the same reference frame $\mathcal{S}$ fixed to the surface of the Earth as in section 8.5 of the lecture notes: $\mathbf{e}_1$ is a unit vector pointing North, $\mathbf{e}_2$ is a unit vector pointing West, and $\mathbf{e}_3$ is a unit vector pointing upwards. The origin of this frame is at a latitude θ on the surface of the Earth.

(a) Including only the effects of a uniform gravitational field and the Coriolis force, show that Newton's second law in the frame $\mathcal{S}$ implies

$$\ddot{\mathbf{r}} \quad \simeq \quad -g\, \mathbf{e}_3 + 2gt\, \boldsymbol{\omega} \wedge \mathbf{e}_3 \ ,$$

where $\boldsymbol{\omega} = \omega \cos\theta\, \mathbf{e}_1 + \omega \sin\theta\, \mathbf{e}_3$ is the angular velocity of the Earth, and we have assumed that terms of order ω^2 are negligible.

(b) Hence show that the trajectory of the particle is given by

$$\mathbf{r} \quad \simeq \quad \left(h - \frac{1}{2}gt^2\right) \mathbf{e}_3 - \frac{1}{3}\omega g t^3 \cos\theta\, \mathbf{e}_2 \ ,$$

and that it lands a distance d to the East of the tower, where

$$d \quad = \quad \frac{1}{3}\omega\sqrt{\frac{8h^3}{g}} \cos\theta \ .$$

Please send comments and corrections to `sparks@maths.ox.ac.uk`.