Analysis II: Continuity and Differentiability   Sheet 2   HT 2020

1. If $f : (0,1] \to \mathbb{R}$ defined by $f(x) = 0$ if x is irrational, and $f(x) = \frac{1}{p+q}$ if $x = p/q \in (0,1]$ where $(p,q) = 1$ (i.e. p and q are co-prime), show that f is continuous at any irrational point in $[0,1]$ and discontinuous at rational numbers in $(0,1]$.

2. Give an example of a continuous function $h : [0,1) \to \mathbb{R}$ which is bounded but does not attain either of its bounds.

3. Assuming the theorem that a continuous real-valued function on a *closed bounded* interval is bounded and attains its bounds, prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(x) \to +\infty$ as $x \to \pm\infty$ then there exists some $x_0 \in \mathbb{R}$ such that $f(x) \geq f(x_0)$ for all $x \in \mathbb{R}$.

4. (*Another proof of IVT*) Let f be a continuous function on a closed interval $[a,b]$, and assume that $f(a) < 0 < f(b)$. Let $\xi = \inf\{t \geq a : f(t) \geq 0\}$ be the *first time* that f hits $[0,\infty)$. Prove that $\xi \in (a,b)$ and $f(\xi) = 0$.

5. (a) Show that every polynomial of odd degree with real coefficients has at least one real root.

(b) Let f be a continuous real-valued function on $[a,b]$. Suppose that B is a real number such that $f(x) \neq B$ for every $x \in [a,b]$, show that $f(x) < B$ for all $x \in [a,b]$ or $f(x) > B$ for all $x \in [a,b]$.

6. How many continuous functions $f : \mathbb{R} \to \mathbb{R}$ are there such that

$$(f(x))^2 = x^2$$

for all $x \in \mathbb{R}$? (Justify your answer.)