Analysis II: Continuity and Differentiability   Sheet 3   HT 2020

1. Let $f : [a, b] \to \mathbb{R}$ be continuous. Suppose that $f(a) < f(b)$ and that f is a 1-1 mapping. Use Intermediate Value Theorem to show that $f(a) < f(x) < f(b)$ for all $x \in (a, b)$. Hence or otherwise prove that f is strictly increasing on $[a, b]$.

2. The function g is defined by

$$g(x) = \frac{x}{1 - |x|} \quad \text{for } -1 < x < 1.$$

Show that g is 1-1, find g^{-1} and determine its domain. Are g and g^{-1} continuous?

3. (a) Which of the following real-valued functions f, defined on $[-1, 1]$ by (i) and (ii) below, have inverses $f^{-1} : [f(-1), f(1)] \to [-1, 1]$? Which have continuous inverses? Given brief reasons.

(i) $f(x) = (x + 1)^2$;

(ii) $f(x) = x$ for $x \in [-1, 0]$ and $f(x) = x + 1$ for $x \in (0, 1]$.

(b) Let $f : (a, b] \to (c, d]$ be strictly increasing and onto, where $a < b$, $c < d$, and b, d are two real numbers. Show that f has a continuous inverse mapping from $(c, d]$ to $(a, b]$.

4. (a) Let $a > 0$. Show that $f(x) = \frac{1}{x}$ is uniformly continuous on $[a, \infty)$.

(b) Show that $f(x) = x^2$ is not uniformly continuous on $[0, \infty)$.

5. (a) Suppose that h is continuous on $[0, \infty)$ and suppose that h is uniformly continuous on $[a, \infty)$ for some positive number a. Show that h is uniformly continuous on $[0, \infty)$.

(b) Show that $f(x) = x^{1/3}$ is uniformly continuous on $\mathbb{R}$. Is it Lipschitz continuous?

6. (a) Let $a < b$ be two real numbers. Suppose that $f : (a, b] \to \mathbb{R}$ is continuous and suppose that the limit of f as $x \to a$ exists. Show that f is uniformly continuous on $(a, b]$.

(b) Suppose now $g : (a, b] \to \mathbb{R}$ is uniformly continuous.

(i) Show that if $(x_n) \subset (a, b]$ is a Cauchy sequence, then $(g(x_n))$ is also a Cauchy sequence.

(ii) Suppose $x_n \in (a, b]$ and $y_n \in (a, b]$ (where $n = 1, 2, \cdots$) are two sequences and $x_n \to a$, $y_n \to a$ as $n \to \infty$. Show that $(g(x_n))$ and $(g(y_n))$ converge to the same limit. Deduce that $g(x)$ has a limit as $x \to a$.