Analysis II: Continuity and Differentiability Sheet 4 HT 2020

1. Consider the following choices of sequence (f_n) of functions on the interval $[0,1]$ whose n^{th} term $f_n(x)$ is defined as follows:
(i) $\left(\frac{x}{2}\right)^n$; (ii) $\sin(nx)$;
(iii) $\frac{x}{1+nx^2}$; (iv) $n^2(1-x)x^n$;
(v) $\sqrt{n}(1-x)x^n$; (vi) $\min\{x^{-1}, n\}$ for $x \in (0,1]$, and 0 for $x = 0$.
(a) For each sequence, determine the pointwise limit or show that it fails to exist.
(b) For each sequence which does converge pointwise, establish whether or not convergence is uniform on $[0,1]$.

2. (a) Prove that the following series converge uniformly on the specified set E:
(i) $\sum_{k=0}^{\infty} x^k/k^{100}$, $E = [-1,1]$;
(ii) $\sum_{k=0}^{\infty} x^{2k}/(2k)!$, $E = [-C, C]$ (for fixed $C > 0$);
(iii) $\sum_{k=0}^{\infty} x^k$, $E = \{x \in \mathbb{R} : |x| \le 1 - 10^{-42}\}$.
(b) Suppose that the real power series $\sum_{k=0}^{\infty} c_k x^k$ converges for every $x \in \mathbb{R}$. Prove that the series converges uniformly on $\mathbb{R}$ if and only if there exists $N \in \mathbb{N}$ such that $c_k = 0$ for all $k > N$.

3. (a) Prove that the series

$$\sum_{k=0}^{\infty} \frac{\sin^2(kx)}{1+k^2x^2}$$

converges for each $x \in \mathbb{R}$ and let $f(x)$ be its sum. Prove that the series converges uniformly on $\{x \in \mathbb{R} : |x| \ge \delta\}$ for any given $\delta > 0$. Deduce that f is continuous on $\mathbb{R} \setminus \{0\}$.
(b) Show that f is not continuous at $x = 0$.

4. (a) Let $g : (a,b) \to \mathbb{R}$ be differentiable at $x_0 \in (a,b)$ and $g(x_0) \neq 0$. Prove that $x \to \frac{1}{g(x)}$ is differentiable at x_0 and

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{g(x_0)^2}.$$

(b) Let $f, g : (a,b) \to \mathbb{R}$ be differentiable at $x_0 \in (a,b)$ with $g(x_0) \neq 0$. Prove that $x \to \frac{f(x)}{g(x)}$ is differentiable at x_0 and the derivative is given by the quotient rule:

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

[*Results from the algebra of limits may be assumed.*]

5. (a) Show that the function

$$f(x) := \begin{cases} x^3 \sin(1/x) & \text{if } x > 0\ , \\ 0 & \text{if } x \le 0 \end{cases}$$

is differentiable for all $x \in \mathbb{R}$ and find its derivative.

(b) Calculate $f''(x)$ for $x \neq 0$, and show that $f''(0)$ does not exist.

(c) Construct a function $g : \mathbb{R} \to \mathbb{R}$ for which g'' exists and is continuous but $g'''(0)$ fails to exist.

[*Where they are applicable, you may use the chain rule and algebraic properties of derivatives.*]