Analysis II: Continuity and Differentiability Sheet 5 HT 2020

1. (a) The functions f, g and h are defined for $x \in \mathbb{R}$ by

$$f(x) = x^3 + 1, \quad g(x) = 1 - (x-1)^3, \quad h(x) = \arctan x.$$

Give explicit formulae for the inverse functions of f and g. Sketch the graphs of f, g and h and of their inverses. Determine the points at which these inverses are differentiable.

(b) Show that there exists a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $(f(x))^5 + f(x) + x = 0$ for all $x \in \mathbb{R}$.

[*Hint: If f exists and has an inverse function g, what should be the function g?*]

(c) Suppose that $f : [a, b] \to \mathbb{R}$ is a strictly increasing continuous function which is twice differentiable at $x_0 \in (a, b)$, with $f'(x_0) \neq 0$. Show that the second derivative of the inverse function g at $f(x_0)$ exists and find a formula for it.

2. (a) Suppose that $h : \mathbb{R} \setminus \{a\} \to \mathbb{R}$ and $h(x) \to l$ as $x \to a$. Show that if $l > 0$, then there exists $\delta > 0$ such that $h(x) > 0$ for all x such that $0 < |x - a| < \delta$.

(b) Now suppose that $f : \mathbb{R} \to \mathbb{R}$ has derivative $f'(a)$ at a. Show that if $f'(a) > 0$ then there exists $\delta > 0$ such that $f(x) > f(a)$ for all x such that $a < x < a + \delta$.

(c) Let $g(x) = |f(x)|$ and suppose that $f(a) = 0$. Prove that g is differentiable at $x = a$ if and only if $f'(a) = 0$.

3. (a) By using Rolle's theorem, prove that if $p(x)$ is a polynomial with real coefficients then the equation

$$(x^2 - x)^2 p'''(x) + 6x(2x^2 - 3x + 1)p''(x) + 6(6x^2 - 6x + 1)p'(x) + 12(2x - 1)p(x) = 0$$

has a solution in $(0, 1)$.

[*Hint. Consider the function $f(x) = (x^2 - x)^2 p(x)$.*]

(b) Let $f(x) = (x^2 - 1)^n$. Prove that for $r = 0, 1, \cdots, n$, $f^{(r)}(x)$ is a polynomial whose value is 0 at no fewer than r distincit points of $(-1, 1)$. Hence prove that $f^{(n)}(x)$ is a polynomial of degree n, with distinct roots, all of which lie in $(-1, 1)$.

4. (a) For which real values does the polynomial $f(x) := 1 + x + \cdots + x^{2m-1}$ take the value 0? What can you say about the sign of $f(x)$ as x varies?

(b) Prove that the function

$$g(x) := 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n}$$

has no real roots when n is even. What can you say about the roots of g when n is odd ?

5. (a) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Which of the following statements are true? In each case give either a proof or a counterexample.

(i) If $f(x) \to \infty$ as $x \to 0$ and $\lim_{x\to 0} f(x)g(x) = 0$, then $\lim_{x\to 0} g(x) = 0$.

(ii) If $f(x) \to \infty$ as $x \to 0$ and $\lim_{x\to 0} g(x) = 0$, then $\lim_{x\to 0} f(x)g(x) = 1$.

(iii) If $f(x) \to \infty$ as $x \to 0$ and $\lim_{x\to 0} g(x) = 1$, then $f(x)g(x) \to \infty$ as $x \to 0$.

(b) (i) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that $f(x) \to k$ as $x \to \infty$, and that g is continuous at k. Prove that $g(f(x)) \to g(k)$ as $x \to \infty$.

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions. Suppose that $f(x) \to \infty$ as $x \to \infty$, and that $g(x) \to l$ as $x \to \infty$. Prove that $g(f(x)) \to l$ as $x \to \infty$.

6. Let f, g and h be three functions on $[a, b]$ which are continuous on $[a, b]$ and differentiable on (a, b). Consider the function

$$F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} \qquad \text{for } x \in [a, b] \ .$$

Show that there is a point $\xi \in (a, b)$ such that $F'(\xi) = 0$. Then, by setting $h \equiv 1$, prove the generalized MVT.