Analysis II: Continuity and Differentiability Sheet 7 HT 2020

[*Every time you use L'Hôpital's Rule you should explain why it is applicable.*]

1. Evaluate the following limits by making use of known derivatives, AOL, and sandwiching techniques, as appropriate:
(i) $\lim_{x\to 0} \frac{x^2}{\sinh^2 x}$;
(ii) $\lim_{x\to 0} \frac{\ln(1+x)}{x}$;
(iii) $\lim_{x\to 0} \frac{\sin^4 x}{x^3}$;
(iv) $\lim_{x\to\infty} \frac{x^2}{\sinh x}$.

2. Evaluate the following limits:
(i) $\lim_{x\to 0} \frac{x}{\tan x}$;
(ii) $\lim_{x\to 0} \frac{\ln\cos x}{x^2}$;
(iii) $\lim_{x\to 0} \frac{e^{x^2}-1}{(e^x-1)^2}$;
(iii) $\lim_{x\to 0} \left(\frac{1}{x^2} - \frac{1}{x\sin x}\right)$.

3. Prove L'Hopital's rule at ∞: Suppose $f, g : (a,\infty) \to \mathbb{R}$ are differentiable, with $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$. If $g'(x) \neq 0$ on (a,∞) and $f'(x)/g'(x) \to l$ as $x \to \infty$, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = l.$$

4. (a) Evaluate $\lim_{x\to\infty} \left(1 + \frac{1}{\sqrt{x}}\right)^{\sqrt{x}}$.
(b) Evaluate $\lim_{x\to-\infty} \left(1 + \frac{1}{x}\right)^{x^2}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathbb{R}$ and assume that $f'''(0)$ exists. Prove that

$$\lim_{h\to 0} \frac{4\left(f(h) - f(-h) - 2\left(f(\frac{h}{2}) - f(-\frac{h}{2})\right)\right)}{h^3} = f'''(0) \ .$$

6. Assume that the conditions for the Mean Value Theorem hold for the function $f : [a, a+h] \to \mathbb{R}$, so that for some $\theta \in (0,1)$ we have

$$f(a+h) - f(a) = hf'(a+\theta h) \ .$$

Fix f and a, and for each non-zero h write $\theta(h)$ for a corresponding value of θ. Prove that if $f''(a)$ exists and is non-zero then

$$\lim_{h\to 0} \theta(h) = \frac{1}{2} \ .$$