11. Counting orbits

Let G act on a set S.
We have seen that the orbits of the action of G partition S.
This partition "corresponds" to the equivalence relation on S: $\quad s \sim s' \quad$ iff $\exists g \in G : gs = s'$

Write S/G for the set of equivalence classes, or equivalently, the set of orbits.

Let $S^g := \{ s \in S \mid gs = s \}$ (also denoted $fix(g)$)

The following is known as Burnside's Lemma; it counts the orbits.

Th: If G is finite then $\left| S/G \right| = \frac{1}{|G|} \sum_{g \in G} |S^g|$

proof: Consider $A = \{ (g,s) \in G \times S \mid gs = s \}$.
We count the elements in A in two different ways:

$$|A| = \sum_{g \in G} |\{ s \in S \mid gs = s \}| = \sum_{g \in G} |S^g|$$

$$|A| = \sum_{s \in S} |\{ g \in G \mid gs = s \}| = \sum_{s \in S} |Stab(s)|$$

$$= \sum_{O_i \in S/G} \underbrace{\sum_{s \in O_i} \underbrace{|Stab(s)|}_{= |G|/|O_i|}}_{|G|}$$

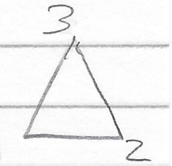
$$= |S/G| \, |G|$$

Addendum: If $g = xhx^{-1}$ then $S^g = S^{xhx^{-1}} = x(S^h)$
Hence $|S^g| = |S^h|$.

So $|S/G| = \frac{1}{|G|} \sum |S^g| \, |C(g)|$

sum over conjugacy classes C_i

$\underline{Ex_1}$: How many ways are there to colour the edges of a triangle with n colours (modular its symmetries)?



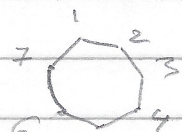
S = set of assigning n colours to "fixed" triangle
$\Rightarrow |S| = n^3$

$G = D_6$ = group of symmetries acting on the triangle and hence on S
$\Rightarrow |G| = 6$

g	$\lvert C(g)\rvert$	$\lvert S^g\rvert$	
e	1	n^3	all as e acts trivially
r	2	n	3 sides need to have the same colour
s	3	n^2	2 " " " "
	$\Sigma\checkmark = 6$		

$\Rightarrow$ number of colourings $= |S/G| = \frac{1}{6}(n^3 + 2\cdot n + 3\cdot n^2)$
$= 1/6\,(n(n+1)(n+2))$

$\underline{Ex_2}$: How many different bracelets are there with 3 red, 2 blue, and 2 white beads?



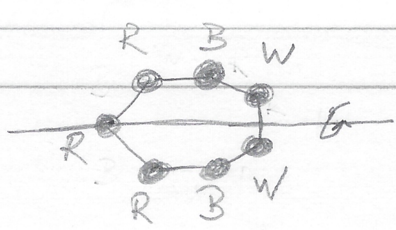
3 red
2 blue
2 white

S = set of arrangements
$\Rightarrow |S| = \frac{7!}{3!2!2!} = 210$

$G = D_{14}$ = group of symmetries of 7-gon
$\Rightarrow |G| = 14$

g	$\lvert C(g)\rvert$	$\lvert S^g\rvert$
e	1	210
r	2	0
r^2	2	0
r^3	2	0
s	7	6
	$\Sigma\checkmark = 14$	

$\Rightarrow$ number of bracelets $=$
$|S/G| = 1/14\,(210 + 7\cdot 6)$
$= 18$



$\underline{Ex_4}$: How many different triples of positive integers are there such that their sum is 100?

$$S = \{ (x_1, x_2, x_3) \in \mathbb{N}^3 \mid x_1 + x_2 + x_3 = 100 \}$$

$$G = S_3 \qquad |S_3| = 6$$

$$|S| = \underbrace{(98)}_{\text{choices for } x_1} \quad \underbrace{(99 - x_1)}_{\text{choices for } x_2} \quad \underbrace{(1)}_{\text{choice for } x_3}$$

$$= \sum_{x_1 = 1}^{98} (99 - x_1) = 98 \cdot 99 - \frac{98 \cdot 99}{2} = 4851$$

g	$\lvert C(g) \rvert$	$\lvert S^g \rvert$
e	1	4851
(123)	2	0
(12)	3	48

$\Sigma = 6$

choices for x_1, then $x_2 = x_3 = \dfrac{100 - x_1}{2}$

number of desired triplets =

$$|S/G| = \frac{1}{6} (4851 + 3 \cdot 48) = 833$$

12. Representations

Th: Every (left) action ρ of a group G on a set S defines a homomorphism

$$\bar{\rho}: G \to \mathrm{Sym}(S), \quad g \mapsto \rho(g,-)$$

and vice versa.

proof: Let G act on S via ρ.

Define $\bar{\rho}(g)$ by the formula $\bar{\rho}(g)(s) = \rho(g,s) = gs$.

By the properties of action $\bar{\rho}(g)$ has inverse $\bar{\rho}(g^{-1})$.

Hence $\bar{\rho}(g) \in \mathrm{Sym}(S)$.

We need to show that $\bar{\rho}(gh) = \bar{\rho}(g) \circ \bar{\rho}(h)$:

$$\forall s \in S: \quad \begin{aligned} \bar{\rho}(gh)(s) &= \rho(gh,s) & &= (gh)s \\ &= \rho(g,\rho(h,s)) & &= g(hs) \\ &= \bar{\rho}(g)(\bar{\rho}(h)(s)) \\ &= (\bar{\rho}(g)\circ\bar{\rho}(h))(s) \end{aligned}$$

So $g \mapsto \rho(g,-)$ is a group homomorphism.

Conversely, given a group homomorphism $\bar{\rho}: G \to \mathrm{Sym}(S)$ define

$$\rho(g,s) = gs := \bar{\rho}(g)(s)$$

Then $\rho(e,s) = \bar{\rho}(e)(s) = id_S(s) = s$

and

$$\begin{aligned} \rho(g,\rho(h,s)) &= \bar{\rho}(g)(\bar{\rho}(h)(s)) \\ &= (\bar{\rho}(g)\circ\bar{\rho}(h))(s) \\ &= \bar{\rho}(gh)(s) \\ &= \rho(gh,s) \end{aligned}$$

Def: G acts on S effectively if $\bar{\rho}$ is injective.

Cayley's Theorem:

Every group G is isomorphic to a subgroup of $Sym(G)$. (set)
In particular,
every finite group G is isomorphic to a subgroup of S_n for $n = |G|$.

Proof: G act on $S = G$ by multiplication

$$\rho(g,x) \stackrel{!}{=} gx$$

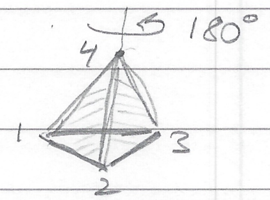
The associated homomorphism

$$\bar{\rho} : G \longrightarrow Sym(G)$$

is injective: $\bar{\rho}(g) = \bar{\rho}(h) \Longleftrightarrow gx = hx \ \forall x \in G \Longleftrightarrow g = h$.
By the First Iso. Th.,

$$G \simeq \operatorname{im} \phi \ \leq \ Sym(G) = S_n \quad \text{with } n = |G|$$

Rotation groups



Tetrahedron:

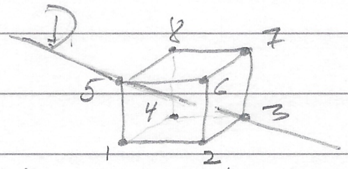
G_T = rotation of a regular tetrahedron
acts on 4 vertices effectively and transitively

Orbit-Stabilizer Formula

$Stab(\text{vertex}) = 3 \text{ rotations} \simeq C_3$

$\Longrightarrow \ |G_T| = |Stab(\text{vertex})| \, |Orb(\text{vertex})| = 3 \cdot 4 = 12$

$\Longrightarrow \ G_T \simeq A_4$



Cube:

G_C = rotations of a cube
acts on 8 vertices effectively and transitively

$Stab(\text{vertex}) = 3 \text{ rotations} \simeq C_3$

$\Longrightarrow \ |G_C| = |Stab(\text{vertex})| \, |Orb(\text{vertex})| = 3 \cdot 8 = 24$

G_C also acts on 4 diagonals effectively and transitively

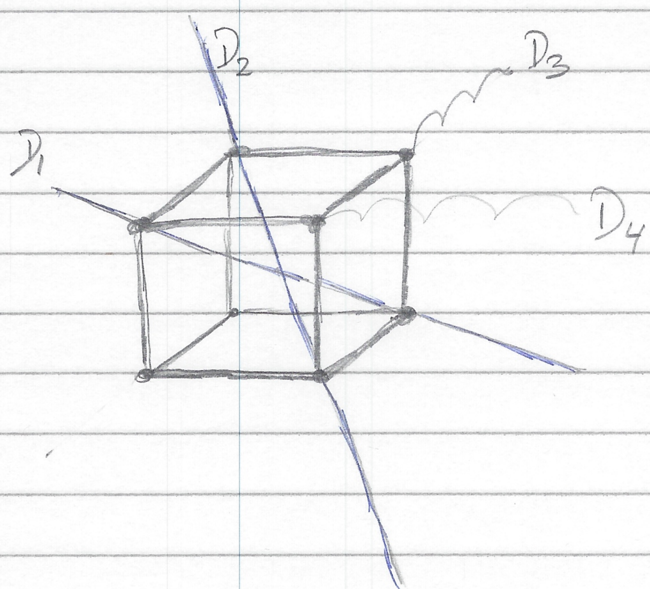
$\Longrightarrow \ G_C \simeq S_4$

Ex: How many ways are there to colour the faces of a cube with n colours?

$|S| = n^6$
$|G_c| = 24$

g	$\|C(g)\|$	rotation	$\|S^g\|$
e	1	identity	n^6
(1234)	6	$\pi/2$ mid faces (top, bottom)	n^3
(123)	8	$-\pi/3$ diagonal D_4	n^2
(12)	6	π mid edges (top left, bottom right)	n^3
$(12)(34)$	3	π mid faces (top, bottom)	n^4

$\Sigma = 24$



way of colouring faces of the cube with n colours =

$$|S/G_c| = \frac{1}{24}\left(n^6 + 6n^3 + 8n^2 + 6n^3 + 3n^4\right)$$