Groups and Group Actions, Sheet 5, TT18

Homomorphisms. Conjugacy. Normal Subgroups.

1. Let G be a group and H a subgroup of G. Show that $gH = Hg$ for all $g \in G$ if and only if $g^{-1}hg \in H$ for all $g \in G$, $h \in H$.

2. Let G be a group and $a \in G$. Show that $C_G(a) = \{g \in G : ag = ga\}$ the *centralizer of a in G*, is a subgroup of G.

Find $C_G(a)$ when (i) $G = S_4$ and $a = (1\,2)(3\,4)$, (ii) $G = A_4$ and $a = (1\,2\,3)$. [Hint: use the fact that $ag = ga$ if and only if $a = g^{-1}ag$.]

3. Recall that the dihedral group D_{2n} (where $n \geqslant 3$) can be defined as

$$D_{2n} = \langle r, s : r^n = e = s^2, sr = r^{-1}s \rangle$$

and that as a set $D_{2n} = \{e, r, \ldots, r^{n-1}, s, rs, \ldots r^{n-1}s\}$. (So r and s are generators of D_{2n} and the rules $r^n = e = s^2$, $sr = r^{-1}s$ are sufficient to completely determine the group table.)

Show, for any integer i, that $sr^i = r^{-i}s$. Also write down each of

$$\left(r^j\right)^{-1} r^i \left(r^j\right), \qquad \left(r^j\right)^{-1} r^i s \left(r^j\right), \qquad \left(r^j s\right)^{-1} r^i \left(r^j s\right), \qquad \left(r^j s\right)^{-1} r^i s \left(r^j s\right),$$

in the form r^k or $r^k s$ for some integer k. Hence determine the conjugacy classes of D_{2n}. You will need to treat separately the cases when n is odd and even.

4. Show that the following maps are homomorphisms. In each case determine the kernel and the image of the homomorphism.

(i) $f_1 : \mathbb{R} \to \mathbb{R}^*$ defined by $f_1(x) = 2^x$.
(ii) $f_2 : \mathbb{C}^* \to \mathbb{R}^*$ defined by $f_2(z) = |z|$.
(iii) $f_3 : S_3 \to S_4$ defined by $f_3(\sigma) = (1\,4)\,\sigma(1\,4)$.
(iv) $f_4 : \mathbb{Z}_n \to \mathbb{C}^*$ defined by $f_4(k) = e^{2\pi i k/n}$.

5. (i) Let G be a group and let ϕ, ψ be automorphisms of G (that is, isomorphisms from G to G). Show that $\phi \circ \psi$ and ϕ^{-1} are automorphisms of G.

Deduce that the set $\mathrm{Aut}(G)$ of automorphisms of G forms a group under composition.

(ii) Given $a \in G$, show that the map $\theta_a : G \to G$ with $\theta_a(g) = aga^{-1}$ is an automorphism of G.

(iii) Show that the map $\Theta : G \to \mathrm{Aut}(G)$ defined by $a \mapsto \theta_a$ is a homomorphism. What is the kernel of Θ?

6. (Optional) (i) Let G be a group and let $\phi : S_3 \to G$ be a homomorphism. Explain why the function ϕ is completely determined by the values of $\phi(12)$ and $\phi(123)$.

(ii) Deduce that there are at most 6 automorphisms of S_3.

(iii) For each $a \in S_3$, determine $\theta_a(12)$ and $\theta_a(123)$.

(iv) Deduce that there *are* 6 automorphisms of S_3 and that $\mathrm{Aut}(S_3)$ is isomorphic to S_3.